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Stochastic Plasticity- A Variational Inequality Formulation and Functional Approximation Approach I: The Linear Case

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Abstract

In this paper we formulate and study the existence and uniqueness of the solution for a class of stochastic mixed variational inequalities arising in problems of infinitesimal elastoplasticity described by uncertain parameters. As a particular example we consider the quasi-static von Mises elastoplastic rate-independent evolution problem with linear elastic behaviour and hardening. For such a problem under the neccessary assumptions we show the equivalency between the variational inequality and a quadratic minimization problem described by a strictly convex, continuous, Gâteaux differentiable, and coercive functional on a Hilbert space. In order to find the unique minimiser we propose the stochastic closest point projection method, obtained by extension of the well known classical return alogorithms to the more general stochastic case. The method is, similarly to its deterministic counterpart, described by non-dissipative and dissipative operators.

Keywords: stochastic variational inequality, plasticity, mixed formulation

Contents

1 Introduction

The uncertainties in inelastic systems arise from a variety of sources including the geometry of the problem, material properties, boundary conditions, initial conditions, or excitations imposed on the system. As a result, depending on the source of randomness, the system behaviour has an uncertain character. In the deterministic sense the parameters describing elastic (reversible)/inelastic (irreversible) behaviour are determined by indentation techniques and then considered as constants in the classical model [32, 19, 14]. However, in a case of heterogenous materials (e.g. soil and bone) such an approach does not properly describe the output due to existence of large variations of uncertainty on the micro-structural level. Thus, in order to give a more reliable description we model material parameters as random fields and processes [3, 22, 23] via maximum entropy principle [33], and reformulate the infinitesimal elastoplastic theory [32] in the stochastic variational setting.

The history of stochastic elastoplasticity begins with the work of Anders and Hori [1] introducing the theory of approximate plasticity based on bounding media analysis. They declared elastic modulus as a source of uncertainty and treated all following subsequent uncertainties with the help of a perturbation technique. Thereafter, Jeremić et al. [31] introduced the Fokker-Plank equation approach based on the work of Kavvas et al. [17], who obtained a generic Eulerian-Lagrangian form of the Fokker-Plank equation corresponding to any nonlinear ordinary differential equation with random forcing and a random coefficient. Recently, in [2] an attempt is made to apply stochastic finite element methods [22, 23, 6] onto stochastic boundary value problems whose formulation involves inequality constraints. However, these methods are either mathematically very complicated to deal with or not enough accurate to be used for. Namely, the perturbation technique is characterised by an inability to accurately approximate the random fields described by moderate and large variances. In addition, the method experiences a "closure-problem" or the dependence of the lower-order moments on the higher-order moments. The moment equations method tries to resolve this situation by direct computation of random solution moments, which leads to the second order exact expression for the evolution of the probability density functions of the stress variable. In this way the closure problem is resolved on the expense of complexity of the algorithm and slight overestimation of the response variance.

The goal of this paper is twofold. First, we extend the mathematical formulation in [14] written with focus on the abstract variational formulation of the elastoplastic problem to the computationally more convenient mixed form. Second, in order to describe the model more realistically we take into account present uncertanties and reformulate the problem into the stochastic variational inequality described by uncertain material parametres in a Hilbert space setting.

The variational formulation as given in [14] concentrates on two alternative dual forms (variational inequalities) of elastoplastic problem, which differ from each other by the form they take, and in the set of variables they use. These problems are referred to as primal and dual formulations. However, the popular radial return solution algorithms as used in engineering practice are requiring the description in which the solution is of dual type, i.e. it consists of primal and dual variable in the same time. Due to this we reformulate the abstract primal variational inequality formulation as given in [14] to a mixed one, by employing the results of the convex analysis [5, 28]. In addition we show that the inequality reduces to a corresponding minimization problem in the stress space solved by a well-posed radial return algorithm [32].

In engineering practice there are many phenomena which may be described by a variational inequality of a particular order, such as for example obstacle [34] and contact [15] problem. Due to the neccesity to solve these problems an powerful mathematical tool has been developed, see e.g. Stampacchia et al. [20, 18], Glowinski et al. [8], Duvaut and Lions [4] etc. However, many of known phenomena have uncertain nature and thus an attempt is made to extend the theory to a more general case described by a stochastic variational inequality (SVI)[11, 10, 12, 7]. By virtue of this theory we pose the mixed SVI for the elastoplastic problem, and study the uniqueness and the existance of the solution by specilizing the general class of random variational inequalities introduced by Gwinner et al. [11, 10, 12]. Similarly as in the deterministic case, we propose and show the well-possedness of the stochastic radial reutrn mapping algorithm in a material point [26, 29, 30]. Particularly we focus on the infinitesimal problem of generalised standard media [13] described by a von Mises yield function. As the variational problem of perfectly plastic materials can not be described within the Sobolev spaces (see [24]) regarding their ability to form shear narrow bands of very high displacement gradients, we restrict ourselves for the sake of simplicity on the problem governed by the Prandtl-Reuss flow rule with linear elastic behaviour and mixed hardening.

The paper is organised as follows: Section 2 overviews the mathematical description of classical deterministic infinitesimal elastoplasticity in concise form. In Section 3 the abstract minimisation formulation is introduced and further employed in Section 4 where the mixed variational form of the abstract problem is derived. The abstract results are specialised in Section 5 where the description of elastoplastic problem in one material point is presented together with the corresponding closest point projection algorithm. By extending the description to one material point Section 6 introduces the mixed variational formulation of deterministic problem with linear mixed hardening plasticity, which is then extended to a stochastic framework in Section 7. Finally, Section 8 concludes the paper.

2 Problem Setting and Motivation

We start with the well known relations for small deformation theory, going from linear elasticity via perfect plasticity to plasticity with linear hardening, which in a stochastic setting is the focus of this paper. The main reason is to exibit the structure of the equations of plasticity based on convex analysis [14], which will be then carried over to the variational inequality formulation for the stochastic problem. This will hopefully show the mathematical similarity between these formulations and thus help to explain the abstract variational formulation and its stochastic interpretation.

2.1 Equilibrium equation

Let us consider a material body occupying a bounded domain $\mathcal{G} \in \mathbb{R}^d$ with a piecewise smooth Lipschitz continuous boundary $\partial\mathcal{G}$ on which are imposed boundary conditions in Dirichlet and Neumann form on $\Gamma_D \subseteq \partial\mathcal{G}$ and $\Gamma_N \subset \partial\mathcal{G}$ respectively, such that $\Gamma_D \cap \Gamma_N = \varnothing$ and $\partial\mathcal{G} = \bar{\Gamma}_N \cup \bar{\Gamma}_D$. The time interval of interest is denoted with $\mathcal{T} = [0, T]$. We focus here on quasi-static small strain associative stochastic elastoplasticity [14, 16, 19, 32] and begin with the equilibrium equation and boundary conditions, which, as all the other relations to follow, are to be understood in a weak sense, to be made precise later:

$$\begin{aligned} -\operatorname{div} \boldsymbol{\sigma}(x) &= \boldsymbol{f}(x) \quad \text{a.e. in } x \in \mathcal{G} \subset \mathbb{R}^d \\ \boldsymbol{\sigma}(x) \cdot \boldsymbol{n}(x) &= \boldsymbol{g}(x), \quad \text{a.e. } x \in \Gamma_N, \\ \boldsymbol{u}(x) &= \boldsymbol{0}, \quad \text{a.e. } x \in \Gamma_D, \end{aligned} \tag{1}$$

where $\boldsymbol{\sigma}(x) \in \operatorname{Sym}(\mathbb{R}^d)$—symmetric tensors on $\mathbb{R}^d$— denotes the stress tensor, $\boldsymbol{f}(x) \in \mathbb{R}^d$ describes volume forces, $\boldsymbol{n}(x) \in \mathbb{R}^d$ denotes the exterior unit normal at $x \in \Gamma_N$, $\boldsymbol{u}(x) \in \mathbb{R}^d$ is the displacement and $\boldsymbol{g}(x) \in \mathbb{R}^d$ is a prescribed surface tension. For the sake of simplicity we use homogenous Dirichlet boundary conditions and under the assumptions of small deformation theory we introduce the strain as the symmetric part of the displacement gradient:

$$\varepsilon(\boldsymbol{u})(x) = \nabla_S \boldsymbol{u}(x) := \frac{1}{2}\left[\nabla \boldsymbol{u}(x) + \nabla \boldsymbol{u}(x)^T\right] \text{ a.e.} \in \mathcal{G}. \tag{2}$$

2.2 Linear elasticity and perfect plasticity

Let us focus on one material point $x \in \mathcal{G}$. In a case of purely linear elastic behaviour when irreversible deformations do not occur, the Helmholtz free energy Eq. (3 a) has a quadratic form in terms of deformations. Due to this the stress is given by Hooke's law Eq. (3 b):

$$\begin{aligned} &\text{a) } \psi_x(\boldsymbol{\varepsilon}_x) = \frac{1}{2}\boldsymbol{\varepsilon}_x : \boldsymbol{A}_x : \boldsymbol{\varepsilon}_x \\ &\text{b) } \boldsymbol{\sigma}_x = \nabla_{\boldsymbol{\varepsilon}}\psi_x = \boldsymbol{A}_x : \boldsymbol{\varepsilon}_x, \end{aligned} \tag{3}$$

where $\boldsymbol{A}_x \in \mathcal{L}(\mathrm{Sym}(\mathbb{R}^d))$ represents the forth order symmetric, bounded, measurable and pointwise-stable elasticity constitutive tensor, i.e., as a linear mapping from $\mathrm{Sym}(\mathbb{R}^d)$ into itself it is symmetric, bounded (uniformly in x) and positive definite (uniformly in x). Here, we have used the abrevation $\boldsymbol{\varepsilon}_x := \boldsymbol{\varepsilon}(x)$ and similarly for other quantities.

Going a step further, in the case of perfect plasticity irreversible changes of shape or size of the body may occur. Thus, one introduces as a measure of irreversible deformation the plastic strain $\boldsymbol{\varepsilon}_{px}$ [14, 16, 32] such that the total deformation is additively decomposed into an elastic $\boldsymbol{\varepsilon}_{ex}$ and plastic part $\boldsymbol{\varepsilon}_{px}$:

$$\boldsymbol{\varepsilon}_x = \boldsymbol{\varepsilon}_{ex} + \boldsymbol{\varepsilon}_{px}, \tag{4}$$

with the plastic deformation $\boldsymbol{\varepsilon}_{px}$ playing the role of an internal variable, i.e., the "memory"of the material. For the sake of simplicity, we consider only isothermic elastoplastic processes described by a Helmholtz free energy as a quadratic function in terms of elastic deformations Eq. (5a), or rather as a function of total strain $\boldsymbol{\varepsilon}_x$ and the '"internal variable" plastic strain $\boldsymbol{\varepsilon}_{px}$. In such case the conjugate thermodynamic force obtains a form of stress given by Eq. (5b), and the elastic domain $\mathcal{K}_x$ — a closed convex set containing the origin which the stress can not leave—describes the evolution of plastic strain according to the associated flow rule. Thus, the state of material is elastic, if the stress $\boldsymbol{\sigma}_x$ is in the interior of $\mathcal{K}_x$, and plastic if the stress belongs to the boundary.

The characterisation of irreversible behaviour in terms of free energy and internal variables allow us to use the thermomechanical description of the plastic state via Clausius-Duhelm formulation of the second law of thermodynamics [16, 32], which requires the entropy production, i.e. dissipation, to be non-negative. Here we require that the dissipation is maximal in the sense that $\dot{\boldsymbol{\varepsilon}}_{px} : \boldsymbol{\sigma}_x \geq \dot{\boldsymbol{\varepsilon}}_{px} : \boldsymbol{\tau}_x$ for all $\boldsymbol{\tau}_x \in \mathcal{K}_x$. As $\mathbf{0} \in \mathcal{K}_x$, the second law is clearly satisfied. Slightly rewritting this statement we arrive at Eq. (5c) which is further discussed.

$$
\begin{aligned}
&\text{a) } \psi_x(\boldsymbol{\varepsilon}_x, \boldsymbol{\varepsilon}_{px}) = \frac{1}{2}(\boldsymbol{\varepsilon}_x - \boldsymbol{\varepsilon}_{px}) : \boldsymbol{A}_x : (\boldsymbol{\varepsilon}_x - \boldsymbol{\varepsilon}_{px}) = \frac{1}{2}\boldsymbol{\varepsilon}_{ex} : \boldsymbol{A}_x : \boldsymbol{\varepsilon}_{ex} \\
&\text{b) } \boldsymbol{\sigma}_x = -\nabla_{\boldsymbol{\varepsilon}_p}\psi_x = -\boldsymbol{A}_x : (\boldsymbol{\varepsilon}_{px} - \boldsymbol{\varepsilon}_x) = \boldsymbol{A}_x : \boldsymbol{\varepsilon}_{ex} \\
&\text{c) } \dot{\boldsymbol{\varepsilon}}_{px} : (\boldsymbol{\tau}_x - \boldsymbol{\sigma}_x) \leq 0, \quad \forall \boldsymbol{\tau}_x \in \mathcal{K}_x
\end{aligned}
\tag{5}
$$

Let us denote the pairing between strain-rates and stress in a more general fashion as a duality pairing $\langle \boldsymbol{\eta}, \boldsymbol{\tau} \rangle_x := \boldsymbol{\eta}_x : \boldsymbol{\tau}_x$, so that relation Eq. (5c) reads:

$$
\langle \dot{\boldsymbol{\varepsilon}}_{px}, \boldsymbol{\tau}_x - \boldsymbol{\sigma}_x \rangle_x \leq 0, \quad \forall \boldsymbol{\tau}_x \in \mathcal{K}_x. \tag{6}
$$

For reasons of material stability, the elastic domain $\mathcal{K}_x$ is a closed convex set and thus Eq. (6) geometricaly means that the plastic flow rate $\dot{\boldsymbol{\varepsilon}}_{px}$ is normal to the boundary of $\mathcal{K}_x$. This condition is known as the normality rule for associated plasticity.

2.2.1 Flow rule

On this example let us recall some notions of convex analysis [14, 21, 28, 5] relevant to this description, which will be used in an abstract way in the theoretical development to follow. We start with two vector spaces in duality, here the space $\mathcal{E}_x$ of strain rates at the material point x and its dual $\mathcal{R}_x$, the space of stresses at x such that the duality pairing Eq. (6) has the physical significance of a dissipation rate. For brevity we drop the index "x" in the following notation. In stress space we are given a closed convex set containing the origin $0 \in \mathcal{K} \subset \mathcal{R}$, the elastic domain which the stress can not leave. What is required is a relation between the thermodynamic forces, here $\boldsymbol{\sigma} \in \mathcal{R}$, and the flux, the rate of the internal variable, $\dot{\boldsymbol{\varepsilon}}_p \in \mathcal{E}$. For associated perfect plasticity this relation is given by Eq. (6). However, convex analysis readily allows some equivalent formulations of Eq. (6), which will be needed later.

For the convex set $\mathcal{K}$ let us define the normal cone at $\boldsymbol{\sigma} \in \mathcal{K}$:

$$
N_{\mathcal{K}}(\boldsymbol{\sigma}) = \{\boldsymbol{\mu} \in \mathcal{E} \mid \langle \boldsymbol{\mu}, \boldsymbol{\tau} - \boldsymbol{\sigma} \rangle \leq 0\} \subseteq \mathcal{E}, \tag{7}
$$

and the indicator function

$$
\Psi_{\mathcal{K}}(\boldsymbol{\sigma}) = \begin{cases} 0 & \text{if } \boldsymbol{\sigma} \in \mathcal{K}, \\ +\infty & \text{otherwise.} \end{cases} \tag{8}
$$

This function is convex as $\mathcal{K}$ is, and lower semi-continuous as $\mathcal{K}$ is closed. Here and elsewhere it makes the theory simpler to consider extended convex functions which may also assume the value $+\infty$. For such a function φ the effective domain is dom $\varphi = \{\boldsymbol{\tau} \mid \varphi(\boldsymbol{\tau}) < +\infty\}$. In a case of a general extended convex function φ the subgradient generalises the notion of gradient such that an element $\boldsymbol{\mu} \in \mathcal{E}$ is a subgradient at $\boldsymbol{\sigma} \in$ dom $\varphi \subseteq \mathcal{R}$ if

$$\forall \boldsymbol{\tau} \in \mathcal{R}: \ \varphi(\boldsymbol{\tau}) \geq \varphi(\boldsymbol{\sigma}) + \langle \boldsymbol{\mu}, \boldsymbol{\tau} - \boldsymbol{\sigma} \rangle. \tag{9}$$

Geometrically this means that the hyperplane defined by $\boldsymbol{\mu}$ passing through $\varphi(\boldsymbol{\sigma})$ is everywhere below the graph of φ. The set of all subgradients is called the subdifferential $\partial\varphi(\boldsymbol{\sigma})$ of φ at $\boldsymbol{\sigma}$:

$$\partial\varphi(\boldsymbol{\sigma}) := \{\boldsymbol{\mu} \in \mathcal{E} \mid \forall \boldsymbol{\tau} \in \mathcal{R}: \ \varphi(\boldsymbol{\tau}) \geq \varphi(\boldsymbol{\sigma}) + \langle \boldsymbol{\mu}, \boldsymbol{\tau} - \boldsymbol{\sigma} \rangle\}. \tag{10}$$

If φ actually has a gradient $\nabla\varphi(\boldsymbol{\sigma})$, then this is the only element in the set $\partial\varphi(\boldsymbol{\sigma}) = \{\nabla\varphi(\boldsymbol{\sigma})\}$. If φ has a minimum at $\boldsymbol{\sigma}$, then $0 \in \partial\varphi(\boldsymbol{\sigma})$, generalising the familiar relation from calculus. It is not difficult to see that Eq. (6) becomes:

$$\dot{\varepsilon}_p \in N_{\mathcal{K}}(\boldsymbol{\sigma}) = \partial\Psi_{\mathcal{K}}(\boldsymbol{\sigma}). \tag{11}$$

A dual formulation would try and specify everything in terms of the strain rate. For this the Fenchel-Legendre transform or conjugate dual of an extended convex function defined on the dual space $\mathcal{E}$ is needed:

$$\varphi^*(\boldsymbol{\mu}) = \sup_{\boldsymbol{\tau} \in \mathcal{R}} \{\langle \boldsymbol{\mu}, \boldsymbol{\tau} \rangle - \varphi(\boldsymbol{\tau})\}, \tag{12}$$

which is also convex and lower semi-continuous. The subgradients of φ and φ^* are related via:

$$\boldsymbol{\mu} \in \partial\varphi(\boldsymbol{\sigma}) \Leftrightarrow \boldsymbol{\sigma} \in \partial\varphi^*(\boldsymbol{\mu}). \tag{13}$$

The conjugate dual of the indicator $\Psi_{\mathcal{K}}$, called the support function of $\mathcal{K}$, is its Fenchel-Legendre transform:

$$\Psi^*_{\mathcal{K}}(\boldsymbol{\mu}) = \sup_{\boldsymbol{\tau} \in \mathcal{R}} \{\langle \boldsymbol{\mu}, \boldsymbol{\tau} \rangle - \Psi_{\mathcal{K}}(\boldsymbol{\tau})\} = \sup_{\boldsymbol{\tau} \in \mathcal{K}} \langle \boldsymbol{\mu}, \boldsymbol{\tau} \rangle. \tag{14}$$

We see from Eq. (6) that $\Psi^*_{\mathcal{K}}(\dot{\boldsymbol{\varepsilon}}_p)$ is the dissipation rate, hence in this context we name

$$j(\dot{\boldsymbol{\varepsilon}}_p) := \Psi^*_{\mathcal{K}}(\dot{\boldsymbol{\varepsilon}}_p) \tag{15}$$

the dissipation function, which is non-negative (as $0 \in \mathcal{K}$), convex, lower-semicontinous, and positively homogenous ($\forall \lambda > 0: j(\lambda\boldsymbol{\mu}) = \lambda j(\boldsymbol{\mu})$) satisfying $j(0) = 0$. For such support functions, one has

$$\mathcal{K} = \partial\Psi^*_{\mathcal{K}}(0) = \partial j(0), \tag{16}$$

and also that (as for any positively homogenous convex function) $\boldsymbol{\sigma} \in \partial j(\boldsymbol{\mu}) \Leftrightarrow \boldsymbol{\sigma} \in \partial j(0)$ and $\langle \boldsymbol{\mu}, \boldsymbol{\sigma} \rangle = j(\boldsymbol{\mu})$. The effective domain of j is:

$$\text{dom } j = \{\boldsymbol{\mu} \mid \langle \boldsymbol{\mu}, \boldsymbol{\tau} \rangle < \infty \quad \forall \boldsymbol{\tau} \in \mathcal{K}\} =: \mathcal{K}^{\infty} \subset \mathcal{E} \tag{17}$$

the so-called barrier cone of $\mathcal{K}$, a closed convex cone. The positive homogenety of j or equivalently the fact that $\partial j^* = \partial \Psi_{\mathcal{K}} = N_{\mathcal{K}}$ is a cone are equivalent expresions of rate independence. The description of the material behaviour can equally well be given in terms of the dissipation function j [32, 16, 14], which is a pseudo-potential according to Eq. (25b), and which allows the elastic domain to be recovered via Eq. (16): $\boldsymbol{\sigma} \in \partial j(\dot{\varepsilon}_p)$.

As for the characterisation of the elastic domain, the most common one is still missing, namely in terms of a yield function. For that the notion of a Minkowski or gauge functional of a convex set $\mathcal{K}$ is needed:

$$g_{\mathcal{K}}(\boldsymbol{\sigma}) = \inf \{\lambda > 0 \mid \boldsymbol{\sigma} \in \lambda \mathcal{K}\}, \tag{18}$$

where $\lambda\mathcal{K} = \{\lambda\boldsymbol{\sigma} \mid \ll \ \boldsymbol{\sigma} \in \mathcal{K}\}$. With the support function or dissipation function $\Psi^*_{\mathcal{K}} = j$ this may be formulated as:

$$g_{\mathcal{K}}(\boldsymbol{\sigma}) = \inf \{\lambda > 0 \mid \forall \boldsymbol{\mu} : \ \langle \boldsymbol{\mu}, \boldsymbol{\sigma} \rangle \leq \lambda j(\boldsymbol{\mu}) \ \} \tag{19}$$

or

$$g_{\mathcal{K}}(\boldsymbol{\sigma}) = \sup_{\boldsymbol{\mu} \neq 0} \frac{\langle \boldsymbol{\mu}, \boldsymbol{\sigma} \rangle}{j(\boldsymbol{\mu})}, \tag{20}$$

giving $g_{\mathcal{K}}(\boldsymbol{\sigma}) j(\boldsymbol{\mu}) \geq \langle \boldsymbol{\mu}, \boldsymbol{\sigma} \rangle$. The set $\mathcal{K}$ may now be characterised by $\mathcal{K} = \{\boldsymbol{\sigma} \mid \rho_{\mathcal{K}}(\boldsymbol{\sigma}) \leq 1\}$. This function is known as the polar function of the support function [14], defined by relations Eq. (19) and Eq. (20), and denoted by:

$$g_{\mathcal{K}}(\boldsymbol{\sigma}) := (\Psi^*_{\mathcal{K}})^o(\boldsymbol{\sigma}) = j^o(\boldsymbol{\sigma}). \tag{21}$$

Observe that:

$$\partial \Psi_{\mathcal{K}}(\boldsymbol{\sigma}) = \bigcup_{\lambda > 0} \lambda \partial g_{\mathcal{K}}(\boldsymbol{\sigma}), \tag{22}$$

where the factor λ has the meaning of a plastic multiplier. The preceding considerations lead immediately to the fact that for any $\boldsymbol{\mu} \in \mathcal{K}^{\infty}$, the set $\mathcal{K}$ is contained in the halfspace:

$$\mathcal{K} \subset H_{\boldsymbol{\mu}} := \{\boldsymbol{\tau} \in \mathcal{R} \mid \langle \boldsymbol{\mu}, \boldsymbol{\tau} \rangle \leq j(\boldsymbol{\mu}) = \Psi^*_{\mathcal{K}}(\boldsymbol{\mu})\}. \tag{23}$$

Finaly, defining the "canonical " yield function:

$$\phi_{\mathcal{K}}(\boldsymbol{\sigma}) := g_{\mathcal{K}}(\boldsymbol{\sigma}) - 1 = j^o(\boldsymbol{\sigma}) - 1 \tag{24}$$

such that $\mathcal{K} = \{\boldsymbol{\sigma} \mid \phi_{\mathcal{K}}(\boldsymbol{\sigma}) \leq 0\}$, one arrives at the following dual equivalent descriptions of the flow rule:

$$
\begin{aligned}
\text{a) } & \dot{\boldsymbol{\varepsilon}}_p \in N_{\mathcal{K}}(\boldsymbol{\sigma}) = \partial \Psi_{\mathcal{K}}(\boldsymbol{\sigma}) = \partial j^*(\boldsymbol{\sigma}), \\
& \Leftrightarrow \forall \boldsymbol{\tau} \in \mathcal{K} : \ \langle \dot{\boldsymbol{\varepsilon}}_p, \boldsymbol{\tau} - \boldsymbol{\sigma} \rangle \leq 0 \\
& \Leftrightarrow \exists \lambda \geq 0 : (\ \dot{\boldsymbol{\varepsilon}}_p \in \lambda \partial \phi_{\mathcal{K}}(\boldsymbol{\sigma})) \wedge (\lambda \phi_{\mathcal{K}}(\boldsymbol{\sigma}) = 0) \\
\text{b) } & \boldsymbol{\sigma} \in \partial j(\dot{\boldsymbol{\varepsilon}}_p) = \partial \Psi^*_{\mathcal{K}}(\dot{\boldsymbol{\varepsilon}}_p) \\
& \Leftrightarrow (\boldsymbol{\sigma} \in \mathcal{K}) \wedge (\langle \dot{\boldsymbol{\varepsilon}}_p, \boldsymbol{\sigma} \rangle = j(\dot{\boldsymbol{\varepsilon}}_p))
\end{aligned} \tag{25}
$$

Similarly, the elastic domain (a closed convex set) $\mathcal{K}$ itself may be characterised in the following equivalent ways as:

$$\mathcal{K} = \{\boldsymbol{\sigma} \mid \phi_{\mathcal{K}}(\boldsymbol{\sigma}) \leq 0\} = \partial j(0) = \bigcap_{\boldsymbol{\mu} \in \mathcal{K}^\infty} H_{\boldsymbol{\mu}}. \tag{26}$$

The last relation becomes important in some numerical approximation schemes. We also see that if the yield function $\phi_{\mathcal{K}}(\boldsymbol{\sigma})$ is smooth with gradient $\nabla \phi_{\mathcal{K}}(\boldsymbol{\sigma})$, then the last line of Eq. (25a) actually gives the familiar relation:

$$\exists \lambda \geq 0 : \ \dot{\boldsymbol{\varepsilon}}_p = \lambda \nabla \phi_{\mathcal{K}}(\boldsymbol{\sigma}) \wedge \lambda \phi_{\mathcal{K}}(\boldsymbol{\sigma}) = 0. \tag{27}$$

The more general formulation in terms of subgradients is very useful in case the yield function is not differentiable, which happens very often even for yield functions which are smooth at a material point when the above descriptions are extended to stress and strain (rate) fields over the whole body.

2.2.2 Time discretisation of the flow rule

Let us divide the time interval $[0, T]$ into steps $\Delta t_n = t_n - t_{n-1}$ with points denoted by t_n in the interval. The goal is to approximate the state of the material such that the relations Eq. (25a) and Eq. (25b) are satisfied at the end of the time increment, given the state of the material at t_{n-1}. This state is described through the values of the total strain $\boldsymbol{\varepsilon}_n$, its incerement $\Delta \boldsymbol{\varepsilon}_n$ and the plastic strain $\boldsymbol{\varepsilon}_{p,n}$ (which then defines the stress $\boldsymbol{\sigma}_n = A : (\boldsymbol{\varepsilon}_n - \boldsymbol{\varepsilon}_{p,n})$). In order to simplify notation, for all quantities to follow, an index "n" is used to denote the quantity at time t_n.

Let us approximate the rate $\dot{\boldsymbol{\varepsilon}}_p$ by the difference quotient $\Delta \boldsymbol{\varepsilon}_{p,n} = (\boldsymbol{\varepsilon}_{p,n} - \boldsymbol{\varepsilon}_{p,n-1})/\Delta t$. In an Euler backward fashion we require this quantity to be in the normal cone at the end of the increment t_n (of Eq. (25a)). This is a special case of Moreau's sweeping process [27]:

$$\frac{1}{\Delta t_n}(\Delta \boldsymbol{\varepsilon}_{p,n}) \in N_{\mathcal{K}}(\boldsymbol{\sigma}_n) = \partial \Psi_{\mathcal{K}}(\boldsymbol{\sigma}_n) \tag{28}$$

As $N_{\mathcal{K}}$ is a cone, it also holds that $\Delta\boldsymbol{\varepsilon}_{p,n} \in N_{\mathcal{K}}(\boldsymbol{\sigma})$ — we utilise rate independence here —and hence the previous equation may be rewritten as a discrete normality rule:

$$\langle \Delta\boldsymbol{\varepsilon}_{p,n}, \boldsymbol{\tau} - \boldsymbol{\sigma}_n \rangle \leq 0, \quad \forall \boldsymbol{\tau} \in \mathcal{K}. \tag{29}$$

2.2.3 Closest point return algorithm

Both because this is the prototype for the actual computation, and as this procedure is used in the abstract proofs, we describe here the well-known return mapping algorithm [32, 16] starting with the Eq. (29). As $\Delta\boldsymbol{\varepsilon}_{p,n} = \boldsymbol{\varepsilon}_{p,n} - \boldsymbol{\varepsilon}_{p,n-1} = \boldsymbol{\varepsilon}_n - \boldsymbol{\varepsilon}_{e,n} - \boldsymbol{\varepsilon}_{n-1} + \boldsymbol{\varepsilon}_{e,n-1} = \Delta\boldsymbol{\varepsilon}_n + \boldsymbol{\varepsilon}_{e,n-1} - \boldsymbol{\varepsilon}_{e,n} = \boldsymbol{A}^{-1} : (\boldsymbol{\sigma}^{trial} - \boldsymbol{\sigma}_n)$ with $\boldsymbol{\sigma}^{trial} = \boldsymbol{A}(\Delta\boldsymbol{\varepsilon}_n + \boldsymbol{\varepsilon}_{e,n-1})$, one obtains from Eq. (29) the variational inequality Eq. (30a) and the equivalent minimisation functional $\mathcal{I}(\boldsymbol{\sigma})$ in Eq. (30b) as a special case of a general variational inequality described in Section 3. This hence leads to a constrained minimisation problem Eq. (30 c) in its familiar "closest-point-return" form. It means that $\boldsymbol{\sigma}_n$ is the projection of $\boldsymbol{\sigma}^{trial}$ onto the closed convex set $\mathcal{K}$ in the metric given by $\boldsymbol{A}^{-1}$, i.e. the norm $\langle \boldsymbol{\sigma} : \boldsymbol{A}^{-1}\boldsymbol{\sigma}\rangle^{1/2}$. Observe that $\boldsymbol{\sigma}^{trial}$ is the stress which would result if the increments were purely elastic.

$$\begin{aligned}
&\text{a) } \langle \boldsymbol{\sigma}_n, \boldsymbol{A}^{-1} : (\boldsymbol{\tau} - \boldsymbol{\sigma}_n)\rangle \geq \langle \boldsymbol{\sigma}^{trial}, \boldsymbol{A}^{-1} : (\boldsymbol{\tau} - \boldsymbol{\sigma}_n)\rangle = \langle \boldsymbol{\varepsilon}_n - \boldsymbol{\varepsilon}_{p,n-1}, \boldsymbol{\tau} - \boldsymbol{\sigma}_n\rangle \\
&\text{b) } \boldsymbol{\sigma}_n = \arg\min_{\boldsymbol{\sigma}\in\mathcal{K}} \mathcal{I}(\boldsymbol{\sigma}) = \arg\min_{\boldsymbol{\sigma}\in\mathcal{K}} [\frac{1}{2}\langle \boldsymbol{\sigma}, \boldsymbol{A}^{-1} : \boldsymbol{\sigma}\rangle - \langle \boldsymbol{\sigma}^{trial}, \boldsymbol{A}^{-1} : \boldsymbol{\sigma}\rangle] \\
&\text{c) } \boldsymbol{\sigma}_n = \arg\min_{\boldsymbol{\sigma}\in\mathcal{K}} \frac{1}{2}\langle \boldsymbol{\sigma}^{trial} - \boldsymbol{\sigma}, \boldsymbol{A}^{-1} : (\boldsymbol{\sigma}^{trial} - \boldsymbol{\sigma})\rangle.
\end{aligned} \tag{30}$$

From this follows the typical operator split of the closest point projection algorithm. First is a reversible, purely elastic step giving $\boldsymbol{\sigma}^{trial}$. If $\boldsymbol{\sigma}^{trial}$ is in the elastic domain $\mathcal{K}$, the minimisation given in Eq. (30b) or Eq. (30c) is trivial as $\boldsymbol{\sigma}_n = \boldsymbol{\sigma}^{trial}$ and the step is reversible purely elastic. In the case $\boldsymbol{\sigma}^{trial} \notin \mathcal{K}$, this is followed by an irreversible purely plastic step of projecting $\boldsymbol{\sigma}^{trial}$ onto $\mathcal{K}$.

2.3 General Hardening

Even though perfect plasticity is in some sense the simplest model of elasto-plastic behaviour it leads to a more complicated analytical situation for the whole body [25], and so we want to consider a more easily tractable case obtained by inclusion of material hardening to the previous process which then turns out to be analytically simpler. Therefore we have an additional internal variable $\boldsymbol{\eta} \in \mathbb{R}^d$ describing hardening, which together with the plastic strain defines the generalised plastic strain denoted by $\boldsymbol{E}_p = (\boldsymbol{\varepsilon}_p, \boldsymbol{\eta})$ at a material point x. The corresponding generalised stress $\boldsymbol{\Sigma} = (\boldsymbol{\sigma}, \boldsymbol{\chi})$ then has for componets the stress $\boldsymbol{\sigma}$ and conjugate thermodynamic force $\boldsymbol{\chi} \in (\mathbb{R}^d)^*$. With this notation, the elastic domain $\mathcal{K}$ becomes a closed convex set containing the origin in the space of $\boldsymbol{\Sigma}$-variables. Following this, one may extend the description given by Eq. (5) to:

$$\begin{aligned} &\text{a) } \psi(\boldsymbol{\varepsilon}, \boldsymbol{E}_p) = \frac{1}{2}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_p) : \boldsymbol{A} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_p) + \psi_{irr}(\boldsymbol{E}_p), \\ &\text{b) } \boldsymbol{\Sigma} = -\nabla_{\boldsymbol{E}_p}\psi, \end{aligned} \tag{31}$$

where the part of the free energy due to the internal variables $\boldsymbol{E}_p$ is denoted by $\psi_{irr}(\boldsymbol{E}_p)$. For the sake of simplicity we take $\psi_{irr}(\boldsymbol{E}_p)$ as a quadratic function $\psi_{irr}(\boldsymbol{E}_p) = \frac{1}{2}\langle \boldsymbol{E}_p, \boldsymbol{H}\boldsymbol{E}_p\rangle$, where $\boldsymbol{H}$ is a symmetric positive definite linear map and the duality pairing $\langle ., .\rangle$ interpreted as $\langle \boldsymbol{\Sigma}, \dot{\boldsymbol{E}}_p\rangle := \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}_p + \langle \boldsymbol{\chi}, \dot{\boldsymbol{\eta}}\rangle$. In component form the relation Eq. (31b) then reads $\boldsymbol{\sigma} = \boldsymbol{A}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_p)$ and $\boldsymbol{\chi} = -\boldsymbol{H}\boldsymbol{\eta}$. Similarly, one may extend the definition of the dissipation function to $j(\dot{\boldsymbol{E}}_p) := \sup(\langle \hat{\boldsymbol{\Sigma}}, \dot{\boldsymbol{E}}_p\rangle, \mid \hat{\boldsymbol{\Sigma}} \in \mathcal{K})$, and require the generalised stress $\boldsymbol{\Sigma}$ to stay in the convex set $\mathcal{K}$. Thus, the abstract mathematical structure remains the same as in perfect plasticity case, only the normality rule (cf. Eq. (25)) generalises to:

$$\begin{aligned} &\text{a) } \dot{\boldsymbol{E}}_p \in N_{\mathcal{K}}(\boldsymbol{\Sigma}) = \partial\Psi_{\mathcal{K}}(\boldsymbol{\Sigma}), \\ &\text{b) } \langle \dot{\boldsymbol{E}}_p, \boldsymbol{T} - \boldsymbol{\Sigma}\rangle \leq 0, \\ &\text{c) } \boldsymbol{\Sigma} \in \partial j(\dot{\boldsymbol{E}}_p) = -\mathbf{A}\boldsymbol{E}_p, \text{ with } \mathbf{A} = \operatorname{diag}(\boldsymbol{A}, \boldsymbol{H}), \end{aligned} \tag{32}$$

valid for generalised standard materials [16].

2.3.1 Time discretisation of the flow rule

As in Section 2.2.3 one may discretise the time interval into n steps and approximate the state of the material such that the relations Eq. (32) are satisfied. Using the same procedure as in case of perfect plasticity this leads to an implicitly discretised general Moreau's sweeping process [27]:

$$\frac{1}{\Delta t_n}(\Delta \boldsymbol{E}_{p,n}) \in N_{\mathcal{K}}(\boldsymbol{\Sigma}_n) = \partial\Psi_{\mathcal{K}}(\boldsymbol{\Sigma}_n). \tag{33}$$

2.3.2 Closest point radial return algorithm

The closest point projection arising from Eq. (33) is a natural extension of the algorithm given by Eq. (30). Taking the generalised increment $\Delta \boldsymbol{E}_n := (\Delta \boldsymbol{\varepsilon}_n, \mathbf{0})$ instead of $\Delta \boldsymbol{\varepsilon}_n$ one may compute the trial stress $\boldsymbol{\Sigma}^{trial} = (\boldsymbol{\sigma}^{trial}, -\boldsymbol{H}\boldsymbol{\eta}_{p,n-1})$ describing purely elastic behaviour. The rest of procedure is essentialy the same as in Section 2.2.3, with the only difference that the variational inequality in Eq. (30a) and minimisation functional in Eq. (30b,c) are expressed in terms of generalised stress instead in terms of Cauchy stress as in case of perfect plasticity.

$$
\begin{aligned}
&\text{a) } \langle \triangle \boldsymbol{E}_{pn}, \boldsymbol{T} - \boldsymbol{\Sigma}_n \rangle \leq 0, \quad \forall \boldsymbol{T} \in \mathcal{K} \\
&\text{b) } \langle \boldsymbol{\Sigma}_n, \mathbf{A}^{-1}(\boldsymbol{T} - \boldsymbol{\Sigma}_n) \rangle \geq \langle \mathbf{A}^{-1}\boldsymbol{\Sigma}^{trial}, \boldsymbol{T} - \boldsymbol{\Sigma}_n \rangle, \quad \forall \boldsymbol{T} \in \mathcal{K} \\
&\text{c) } \boldsymbol{\Sigma}_n = \arg \min_{\boldsymbol{\Sigma} \in \mathcal{K}} \mathcal{I}(\boldsymbol{\Sigma}) = \arg \min_{\boldsymbol{\Sigma} \in \mathcal{K}} [\frac{1}{2}\langle \mathbf{A}^{-1}\boldsymbol{\Sigma}, \boldsymbol{\Sigma} \rangle - \langle \mathbf{A}^{-1}\boldsymbol{\Sigma}^{trial}, \boldsymbol{\Sigma} \rangle] \\
&\text{d) } \boldsymbol{\Sigma}_n = \arg \min_{\boldsymbol{\Sigma} \in \mathcal{K}} \{\frac{1}{2}\langle \mathbf{A}^{-1}(\boldsymbol{\Sigma}^{trial} - \boldsymbol{\Sigma}), \boldsymbol{\Sigma}^{trial} - \boldsymbol{\Sigma} \rangle\}
\end{aligned}
\tag{34}
$$

2.4 Linear isotropic and kinematic hardening

For the sake of simplicity we consider a special case of general hardening (Section 2.3) namely linear isotropic and kinematic hardening. This means that the generalised stress consists of the Cauchy stress $\boldsymbol{\sigma}$ and thermodynamic force $\boldsymbol{\chi} = \{\boldsymbol{\varsigma}, \zeta\}$ compensed of the backstress $\boldsymbol{\varsigma}$ (kinematic part) and the conjugate force ζ (isotropic part). Its energy conjugated internal variable $\boldsymbol{E}_p$ then turns out to be the pair $(\boldsymbol{\varepsilon}_p, \nu)$ consisting of plastic strain $\boldsymbol{\varepsilon}_p$ and a scalar measure ν of the diameter of the elastic domain. These measures describe the kinematic and isotropic hardening phenomena. Additionaly, the generalised constitutive tensor $\boldsymbol{H}$ reduces to $\mathrm{diag}(H_{iso}, \boldsymbol{H}_{kin})$ where $H_{iso}(x)$ and $\boldsymbol{H}_{kin}(x)$ are the isotropic and kinematic constitutive tensors, uniformly bounded and positive definite almost everywhere on $\mathcal{G}$. Moreover, the irreversible energy Eq. (31a) and the stress law Eq. (31b) reduce to

$$
\begin{aligned}
&\text{a) } \psi_{irr}(\boldsymbol{E}_p) = \frac{1}{2}H_{iso}\nu^2 + \frac{1}{2}\boldsymbol{H}_{kin}\boldsymbol{\varepsilon}_p : \boldsymbol{\varepsilon}_p. \\
&\text{b) } \boldsymbol{\varsigma} = -\boldsymbol{H}_{kin} : \boldsymbol{\varepsilon}_p, \quad \zeta = -H_{iso}\nu.
\end{aligned}
\tag{35}
$$

which further allow us to directly apply the results of Section 2.3 to this particular case.

3 Minimisation of a quadratic functional

After the preceding well-known description at a material point, the present and next section will cover some abstract results. These then apply not only to the material point description, but to the whole deterministic and stochastic plasticity problem.

Let $\Phi(z)$ be a strictly convex, continuous, Gâteaux differentiable, and coercive functional on a Hilbert space $\mathcal{Z}$, i.e. $\Phi(z) \to \infty$ as $\|z\| \to \infty$. In particular we look at a continuous (or bounded $a(z_1, z_2) \leq c\|z_1\|\|z_2\|$), symmetric and $\mathcal{Z}$-elliptic ($a(z,z) \geq c\|z\|^2$) bilinear form $a : \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}$ and an element $y \in \mathcal{Z}^*$. These we use to define the functional:

$$\Phi(z) = \frac{1}{2}a(z,z) - \langle y, z\rangle. \tag{36}$$

As a and y are continuous and Gâteaux-differentiable, and as a is $\mathcal{Z}$-elliptic, Φ has all the properties stated above. To handle the dissipation we have to allow for a second convex functional j on $\mathcal{Z}$ which may not be Gâteaux differentiable everywhere. This functional is supposed to be the support functional of a closed convex set $\mathcal{K} \subset \mathcal{Z}^*$ containing the origin. We then have (see [14]):

Proposition 3.1. *With the notation and assumptions just described, the problem to minimise:*

$$\min_{z \in \mathcal{Z}} \left(\Phi(z) + j(z)\right) \tag{37}$$

has a unique solution:

$$w = \arg\min_{z \in \mathcal{Z}} \left(\Phi(z) + j(z)\right), \tag{38}$$

characterised by $0 \in \partial(\Phi(w) + j(w))$*, i.e.*

$$-\delta\Phi(w) \in \partial j(w), \tag{39}$$

where $\delta\Phi(w) = a(w,\cdot) - y$ *is the Gâteaux derivative of* Φ*. The last relation may also be written as:*

$$\forall z \in \mathcal{Z} : a(w, z-w) + j(z) - j(w) \geq \langle y, z-w\rangle, \tag{40}$$

i.e. an elliptic variational inequality of the second kind.

Proof. See [9] □

Let us for brevity denote $w^* = -\delta\Phi(w)$. Then Eq. (39) becomes

$$w^* \in \partial j(w), \tag{41}$$

which is equivalent to (see Section 2.2)

$$w \in \partial j^*(w^*) = \partial\Psi_{\mathcal{K}}(w^*), \tag{42}$$

or

$$\langle w, z^* - w^* \rangle \leq 0, \quad \forall z^* \in \mathcal{K}. \tag{43}$$

Collecting all, we have proven the characterisation of the solution in a mixed variational description:

Theorem 3.2. *With the notation and assumptions as before, the problem:*

$$\min_{z \in \mathcal{Z}} \left(\Phi(z) + j(z) \right) \tag{44}$$

has a unique solution $w \in \mathcal{Z}$*,*

$$w = \arg\min_{z \in \mathcal{Z}} \left(\Phi(z) + j(z) \right) \tag{45}$$

characterised by

$$\exists w^* \in \mathcal{K},\ \forall z \in \mathcal{Z} :\ a(w, z) + \langle w^*, z \rangle = \langle y, z \rangle. \tag{46}$$

and

$$\forall z^* \in \mathcal{K} :\ \langle w, z^* - w^* \rangle \leq 0. \tag{47}$$

The bilinear form a defines a linear, continuous, self-adjoint, and coercive ($\langle Az, z \rangle \geq c_2 \|z\|^2$) operator $A : \mathcal{Z} \to \mathcal{Z}^*$ via

$$\forall v, z \in \mathcal{Z} :\ a(z, v) = \langle Az, v \rangle. \tag{48}$$

Due to the properties just stated, A has an inverse $A^{-1} : \mathcal{Z}^* \to \mathcal{Z}$ with the same attributes. This allows us to define a bilinear, continuous, symmetric and coercive form a^* on $\mathcal{Z}^*$:

$$a^*(z_1^*, z_2^*) = \langle z_1^*, A^{-1} z_2^* \rangle, \tag{49}$$

or in other words if $u \in \mathcal{Z}$ solves:

$$\forall z \in \mathcal{Z} :\ a(u, z) = \langle z_2^*, z \rangle, \tag{50}$$

then

$$a^*(z_1^*, z_2^*) = \langle z_1^*, u \rangle. \tag{51}$$

If the bilinear form a can be identified in our application with the Helmholz free energy, then a^* is the complementary energy. We need the following result, now for variational inequalities of the first kind:

Proposition 3.3. *Let $\mathcal{V}$ be a Hilbert space, $\varphi : \mathcal{Z} \to \mathbb{R}$ a strictly convex Gâteaux-differentiable, coercive functional, and $\mathcal{K} \subset \mathcal{V}$ a non-empty, closed, convex set containg the origin. Then the minimisation problem:*

$$\min_{v \in \mathcal{K}} \varphi(v) \tag{52}$$

has a unique solution $u \in \mathcal{V}$,

$$u = \arg\min_{v \in \mathcal{K}} \varphi(v), \tag{53}$$

characterised by

$$\forall v \in \mathcal{K} : \quad \langle \delta\varphi(u), v - u \rangle \geq 0, \tag{54}$$

where $\delta\varphi$ is the Gâteaux-derivative of φ.

Proof. See [9] □

We shall now take $\mathcal{V} = \mathcal{Z}^*$, and $\varphi(z^*) = \frac{1}{2} a^*(y - z^*, y - z^*)$ with Gâteaux derivative:

$$\delta\varphi(z^*) = a^*(z^*, \cdot) - a^*(y, \cdot) = a^*(z^* - y, \cdot). \tag{55}$$

We see from Eq. (46) in Theorem 3.2 that w solves :

$$\forall z \in \mathcal{Z} : \quad a(w, z) = \langle y - w^*, z \rangle, \tag{56}$$

and hence with Eq. (50)

$$\varphi(w^*) = \frac{1}{2} a^*(y - w^*, y - w^*) = \frac{1}{2} \langle y - w^*, w \rangle, \tag{57}$$

and

$$\delta\varphi(w^*) = a^*(w^* - y, \cdot) = -w. \tag{58}$$

Eq. (54) reads

$$\forall z^* \in \mathcal{K} : \quad -\langle w, z^* - w^* \rangle \geq 0. \tag{59}$$

We collect these results in

Theorem 3.4. *With the notation and assumptions as before, the problem in Theorem 3.2 is equivalent to:*

$$w^* = \arg\min_{z^* \in \mathcal{K}} \frac{1}{2} a^*(y - z^*, y - z^*) \tag{60}$$

(w^* is in $\mathcal{K}$ the closest point to y in the a^* metric), characterised by:

$$\exists w \in \mathcal{Z}, \forall z \in \mathcal{Z} : \quad a(w, z) = \langle y - w^*, z \rangle \tag{61}$$

and

$$\forall z^* \in \mathcal{K} : \quad \langle w, z^* - w^* \rangle \leq 0. \tag{62}$$

This is the abstract formulation of the closest point algorithm outlined in Eq. (30). Hence, computing w^* as the closest point in Eq. (60), the pair (w, w^*) satisfies Theorem 3.2.

4 Variational formulation

The variational formulation for elastoplastic problems includes variational inequalities and corresponding minimisation problems. In this section we collect some general results on this topic which are needed in the sequel.

4.1 Abstract Plasticity Problem

Plasticity describes irreversible evolution, and for the time-discretised problem the results of the previous Section 3 will be used. In order to describe the time-continuous problem of evolution some function spaces are needed. Using the notation of the previous section, for $1 \le p < \infty$ we denote by:

$$L_p(\mathcal{T}, \mathcal{Z}) = \{v : \mathcal{T} \to \mathcal{Z} \mid \|v\|_{L_p} = \left(\int_0^T \|v(t)\|_{\mathcal{Z}}^p \, \mathrm{d}t \right)^{1/p} < \infty\} \tag{63}$$

the space of Bochner-Lebesgue p-integrable $\mathcal{Z}$-valued functions, with the usual extension to $p = \infty$ with:

$$\|v\|_{L_\infty} = \operatorname*{ess\,sup}_{t \in \mathcal{T}} \|v(t)\|_{\mathcal{Z}}. \tag{64}$$

Going a step further, we introduce the Sobolev space:

$$H^1(\mathcal{T}, \mathcal{Z}) = \{v \in L_2(\mathcal{T}, \mathcal{Z}) \mid \|v\|_{H_1} = (\|v\|_{L_2}^2 + \|\dot{v}\|_{L_2}^2)^{\frac{1}{2}} < \infty\}, \tag{65}$$

where $\dot{v} := \frac{d}{dt} v$ is the weak derivative w.r.t. $t \in \mathcal{T}$. Using the notation and assumptions as before one may formulate the first result for an abstract plasticity problem in a primal formulation which we cite from [14]. Observe that $\mathcal{K}^\infty = \operatorname{dom} j \subset \mathcal{Z}$ is the barrier cone of the closed convex set (elastic domain) $\mathcal{K} \subset \mathcal{Z}^*$ with $0 \in \mathcal{K}$, the effective domain of the support functional $j(z) = \Psi_{\mathcal{K}}^*(z)$, the dissipation function.

With this notation, we may rephrase Problem ABS from [14]:

Proposition 4.1. ***Problem ABS-P*** *Given a function $f \in H^1(\mathcal{T}, \mathcal{Z}^*)$ with $f(0) = 0$, there exists a unique function $w \in H^1(\mathcal{T}, \mathcal{Z}^*)$ with $w(0) = 0$ and $\dot{w}(t) \in \mathcal{K}^\infty$, which solves the following problem a.e. in $t \in \mathcal{T}$:*

$$\forall z \in \mathcal{Z} : \quad a(w(t), z - \dot{w}(t)) + j(z) - j(\dot{w}(t)) \ge \langle f(t), z - \dot{w}(t) \rangle. \tag{66}$$

If in addition $f_1, f_2 \in H^1(\mathcal{T}, \mathcal{Z}^*)$ with $f_1(0) = f_2(0)$ are two different loadings, and $w_1, w_2 \in H^1(\mathcal{T}, \mathcal{Z}^*)$ are the corresponding solutions, then

$$\|w_1 - w_2\|_{L_\infty} \leq c\|\dot{f}_1 - \dot{f}_2\|_{L_1}. \tag{67}$$

Proof. See [14]. □

We want to reformulate this result in a mixed form by introducing the function $w^* \in H^1(\mathcal{T}, \mathcal{Z}^*)$

$$w^* = f - a(w, \cdot), \tag{68}$$

so that Eq. (66) may be written as

$$\forall z \in \mathcal{Z}: \; j(z) \geq j(\dot{w}(t)) + \langle w^*(t), z - \dot{w}(t)\rangle, \tag{69}$$

showing that $w^*(t) \in \partial j(\dot{w}(t))$ a.e. $t \in \mathcal{T}$. Hence, this is then equivalent to:

$$\dot{w}(t) \in \partial j^*(w^*(t)) \quad \text{a.e. } t \in \mathcal{T}, \tag{70}$$

which is the same as:

$$\forall z^* \in \mathcal{K}: \quad \langle \dot{w}(t), z^* - w^*(t)\rangle \leq 0 \quad \text{a.e. } t \in \mathcal{T}. \tag{71}$$

Collecting everything, one can show the well-posedness of a mixed formulation of an abstract plasticity problem, the one we will be using:

Theorem 4.2. ***Problem ABS-M.*** *With the notation and assumptions as above, there are unique functions, $w \in H^1(\mathcal{T}, \mathcal{Z})$ and $w^* \in H^1(\mathcal{T}, \mathcal{Z}^*)$ with $w(0) = 0$ and $w^*(0) = 0$, which solve the following problem a.e. $t \in \mathcal{T}$:*

$$\forall z \in \mathcal{Z}: \; a(w(t), z) + \langle w^*(t), z\rangle = \langle f(t), z\rangle \tag{72}$$

and

$$\forall z^* \in \mathcal{K}: \; \langle \dot{w}(t), z^* - w^*(t)\rangle \leq 0. \tag{73}$$

If in addition $f_1, f_2 \in H^1(\mathcal{T}, \mathcal{Z})$ with $f_1(0) = f_2(0) = 0$ are two different loadings and $w_1, w_2 \in H^1(\mathcal{T}, \mathcal{Z})$ and $w_1^*, w_2^* \in H^1(\mathcal{T}, \mathcal{Z}^*)$ the corresponding solutions, then

$$\|w_1 - w_2\|_{L_\infty} \leq c\|\dot{f}_1 - \dot{f}_2\|_{L_1} \tag{74}$$

and

$$\|w_1^* - w_2^*\|_{L_\infty} \leq c^*(\|\dot{f}_1 - \dot{f}_2\|_{L_1} + \|f_1 - f_2\|_{L_\infty}) \leq c^{**}\|\dot{f}_1 - \dot{f}_2\|_{L_1} \tag{75}$$

Proof. Everything except the last estimate Eq. (74) follows from Proposition 4.1 or was already shown. To prove that last estimate, observe that:

$$\forall z: \ \langle w_1^*(t), z\rangle = \langle f_1(t), z\rangle - a(w_1(t), z) \tag{76}$$

$$\text{and } \langle w_2^*(t), z\rangle = \langle f_2(t), z\rangle - a(w_2(t), z). \tag{77}$$

Substracting the second from the first equation and taking for $z \in \mathcal{Z}$ the solution of

$$\forall v \in \mathcal{Z}: \ a(z, v) = \langle w_1^* - w_2^*, v\rangle, \tag{78}$$

together with the boundness and coercivity of a, one obtains a.e. $t \in \mathcal{T}$

$$\|w_1^*(t) - w_2^*(t)\| \leq c(\|f_1(t) - f_2(t)\| + \|w_1(t) - w_2(t)\|, \tag{79}$$

which gives

$$\|w_1^* - w_2^*\|_{L_\infty} \leq c(\|f_1 - f_2\|_{L_\infty} + \|w_1 - w_2\|_{L_\infty}). \tag{80}$$

Using now the estimate Eq. (74), one obtains the first inequality in Eq. (75). As $f_1(0) = f_2(0) = 0$ we have:

$$f_1(t) - f_2(t) = \int_0^t (\dot{f}_1(s) - \dot{f}_2(s))\mathrm{d}s, \tag{81}$$

and hence

$$\|f_1 - f_2\|_{L_\infty} \leq \sup_{t\in\mathcal{T}} \int_0^t \|\dot{f}_1(t) - \dot{f}_2(t)\|\mathrm{d}t \leq c\|\dot{f}_1 - \dot{f}_2\|_{L_1} \tag{82}$$

proving the second inequality in Eq. (75). □

4.2 Time discretisation of the abstract problem

We turn again to the abstract plasticity problem ABS-P Proposition 4.1 and ABS-M Theorem 4.2. In [14] it is shown that problem ABS-P may be discretised by the Euler backward method: divide the interval $[0, T]$ into steps of size $\triangle t$, such that $t_0 = 0$ and $t_n = n\triangle t$ and approximate $\dot{w}(t_n)$ by

$$\dot{w}(t_n) = \dot{w}_n \approx (w(t_n) - w(t_{n-1}))/\triangle t = \triangle w_n/\triangle t. \tag{83}$$

Then the evolutionary variational inequality Eq. (66) may be approximated by the following algorithm [14], which is a special case of Moreau's sweeping process [27]:

Proposition 4.3. *With the same notation and assumptions as before, taking for all the increment $\Delta w_n \in \mathcal{K}^\infty$ as the solution of*

$$\begin{aligned}\forall z \in \mathcal{Z}: \quad & a(\Delta w_n, z - \Delta w_n) + j(z) - j(\Delta w_n) \geq \\ & \qquad \langle f(t_n), z - \Delta w_n \rangle - a(w_{n-1}, z - \Delta w_n)\end{aligned} \tag{84}$$

the approximate solution $(w_0, w_1, ..., w_n, ..)$ converges to the solution $w(t)$ of problem ABS-P in Proposition 4.1 as $\Delta t \to 0$.

Proof. See [14]. □

With the notation $y_n = f(t_n) - a(w_{n-1}, \cdot)$, we see that Eq. (84) is obviously of the type given in Section 3 in Proposition 3.1, and hence has a unique solution Δw_n from a corresponding minimisation problem. Putting this in a mixed variational formulation, we obtain from Proposition 4.3 and Theorem 3.2 a time-discrete version of Theorem 4.2:

Theorem 4.4. *with the notation as before the problem in Proposition 4.3 has a unique solution $\Delta w_n \in \mathcal{K}^\infty \subset \mathcal{Z}$, characterised by: $\forall t_n, \exists w_n^* \in \mathcal{K}$ such that*

$$\forall z \in \mathcal{Z}: \; a(\Delta w_n, z) + \langle w_n^*, z \rangle = \langle y_n, z \rangle \tag{85}$$

and

$$\forall z^* \in \mathcal{K}: \; \langle \Delta w_n, z^* - w_n^* \rangle \leq 0. \tag{86}$$

The appoximate solutions $\{w_n\}$, $\{w_n^*\}$ converge as $\Delta t \to 0$ to the solutions $w(t)$ and $w^*(t)$ of problem ABS-M in Theorem 4.2.

Proof. Everything except the convergence of w_n^* is already shown. This follows along analogous arguments as the proof of uniqueness in Theorem 4.2. □

To obtain the analogue of Theorem 3.4, we collect all results to formulate:

Theorem 4.5. ***Problem ABS-D*** *With the notation and assumptions as before, the computation of a unique Δw_n, w_n^* in Theorem 4.4 can be performed by*

$$w_n^* = \arg\min_{z^* \in \mathcal{K}} \frac{1}{2} a^*(y_n - z^*, y_n - z^*) \tag{87}$$

with $y_n = f(t_n) - a(w_{n-1}, \cdot)$ and w_n^* being the closest point in $\mathcal{K}$ to y_n in the a^*-metric. Computing $\Delta w_n \in \mathcal{Z}$ by

$$\forall z \in \mathcal{Z}: \; a(\Delta w_n, z) = \langle y_n - w_n^*, z \rangle \tag{88}$$

one has $\Delta w_n \in \mathcal{K}^\infty$, which satisfies

$$\forall z^* \in \mathcal{K}: \; \langle \Delta w_n, z^* - w_n^* \rangle \leq 0. \tag{89}$$

5 Plasticity problem at a material point

With the results established in the previous Section 5, one may show that the problems from Section 3 are well-posed. We here start with perfect plasticity and then continue with the more complex cases.

With the notation of Section 2.2, assume that a strain evolution $\boldsymbol{\varepsilon}_x \in H^1(\mathcal{T}, \mathrm{Sym}(\mathbb{R}^d)\}$ with $\boldsymbol{\varepsilon}_x(0) = 0$ is given at material point x. Define $\boldsymbol{\zeta}_x(t) = A_x \boldsymbol{\varepsilon}_x(t) \in H^1(\mathcal{T}, \mathrm{Sym}(\mathbb{R}^d))$, and the symmetric and coercive bilinear form:

$$a^p(\boldsymbol{\varepsilon}_x^1, \boldsymbol{\varepsilon}_x^2) = \boldsymbol{\varepsilon}_x^1 : A_x : \boldsymbol{\varepsilon}_x^2. \tag{90}$$

To use Theorem 4.2, we identify $\mathcal{Z} = \mathcal{Z}^* = \mathrm{Sym}(\mathbb{R}^d)$ and set $w = \boldsymbol{\varepsilon}_{px}$ and $w^* = \boldsymbol{\sigma}_x$. Then from Theorem 4.2 one obtains

Corollary 5.1. *There are unique functions $\boldsymbol{\varepsilon}_{px} = H^1(\mathcal{T}, Sym(\mathbb{R}^d))$ and $\boldsymbol{\sigma}_x \in H^1(\mathcal{T}, Sym(\mathbb{R}^d))$ such that a.e. $t \in \mathcal{T}$:*

$$\forall \boldsymbol{\mu}_x \in Sym(\mathbb{R}^d): \quad a^p(\boldsymbol{\varepsilon}_{px}(t), \boldsymbol{\mu}_x) + \langle \boldsymbol{\sigma}_x(t), \boldsymbol{\mu}_x \rangle_x = \langle \boldsymbol{\zeta}_x(t), \boldsymbol{\mu}_x \rangle_x \tag{91}$$

$$\text{and } \forall \boldsymbol{\tau}_x \in \mathcal{K}_x : \quad \langle \dot{\boldsymbol{\varepsilon}}_{px}(t), \boldsymbol{\tau}_x - \boldsymbol{\sigma}_x(t) \rangle_x \le 0. \tag{92}$$

Proof. Existence, uniqueness, and continuous dependence now follow from Theorem 4.2. □

With the notation of Section 2.4, assume that a strain evolution $\boldsymbol{\varepsilon}_x \in H^1(\mathcal{T}, \mathrm{Sym}(\mathbb{R}^d))$ with $\boldsymbol{\varepsilon}_x(0) = 0$ is given. Define $\boldsymbol{S}_x \in H^1(\mathcal{T}, \mathrm{Sym}(\mathbb{R}^d) \times \mathbb{R}^m)$ by

$$\boldsymbol{S}_x(t) = (\boldsymbol{A}_x : \boldsymbol{\varepsilon}_x(t), \boldsymbol{0}), \tag{93}$$

and the symmetric and coercive bilinear form a^g on $\mathrm{Sym}(\mathbb{R}^d) \times \mathbb{R}^m$ by:

$$a^g(\boldsymbol{E}_{px}^1, \boldsymbol{E}_{px}^2) = a^p(\boldsymbol{\varepsilon}_{px}^1, \boldsymbol{\varepsilon}_{px}^2) + \langle \boldsymbol{\eta}_x^1, \boldsymbol{H}\boldsymbol{\eta}_x^2 \rangle_x, \tag{94}$$

where $\boldsymbol{E}_p^k := (\boldsymbol{\varepsilon}_{px}^k, \boldsymbol{\eta}_x^k)$. Now identify $\mathcal{Z} = \mathcal{Z}^* = \mathrm{Sym}(\mathbb{R}^d) \times \mathbb{R}^m$, and set $w = \boldsymbol{E}_{px}$ and $w^* = \boldsymbol{\Sigma}_x$. Then Theorem 4.2 implies:

Corollary 5.2. *There are unique functions $\boldsymbol{E}_{px} \in H^1(\mathcal{T}, Sym(\mathbb{R}^d) \times \mathbb{R}^m)$ and $\boldsymbol{\Sigma}_x \in H^1(\mathcal{T},\ Sym(\mathbb{R}^d) \times \mathbb{R}^m)$ such that a.e. $t \in \mathcal{T}$:*

$$\forall \boldsymbol{M}_x \in Sym(\mathbb{R}^d) \times \mathbb{R}^m : \quad a^g(\boldsymbol{E}_{px}(t), \boldsymbol{M}_x) + \langle \boldsymbol{\Sigma}_x(t), \boldsymbol{M}_x \rangle_x = \langle \boldsymbol{S}_x(t), \boldsymbol{M}_x \rangle_x \tag{95}$$

and

$$\forall \boldsymbol{T}_x \in \mathcal{K}_x : \langle \dot{\boldsymbol{E}}_{px}(t), \boldsymbol{T}_x - \boldsymbol{\Sigma}_t(t) \rangle_x \le 0. \tag{96}$$

Proof. Existence, uniqueness and continuous dependence follow from Theorem 4.2. □

Now it is no problem to specialise this result to the description of combined isotropic and kinematic hardening in Section 2.4. We identify $\boldsymbol{\eta}_x = (\boldsymbol{\varepsilon}_{px}, \nu_x)$ and set

$$\langle \boldsymbol{E}_{px}, \boldsymbol{H}_x \boldsymbol{E}_{px} \rangle_x = \boldsymbol{H}_{kin,x} \boldsymbol{\varepsilon}_{px} : \boldsymbol{\varepsilon}_{px} + H_{iso,x} \nu_x \cdot \nu_x. \tag{97}$$

With this we may define the symmetric and coercive bilinear form a^h by

$$a^h(\boldsymbol{E}^1_{px}, \boldsymbol{E}^2_{px}) = a^p(\boldsymbol{\varepsilon}^1_{px}, \boldsymbol{\varepsilon}^2_{px}) + \boldsymbol{H}_{kin,x} \boldsymbol{\varepsilon}^1_{px} : \boldsymbol{\varepsilon}^2_{px} + H_{iso,x} \nu^1_x \cdot \nu^2_x. \tag{98}$$

Corollary 5.2 yields the well-possedness of this special combined hardening plasticity problem:

Corollary 5.3. *The case of combined isotropic and kinematic hardening is a special case of the problem with general hardening, hence the same conditions hold and the combined hardening problem is well-posed.*

With this we have now shown the well-posedness of all time-continous problems described at a material point x in Section 3.

5.1 Time discretisation at a material point

We now apply Theorem 4.4 to the problems treated in Section 5. We use the same identification of $\mathcal{Z}, \mathcal{Z}^*$ and variables as in Section 5 and the time discretisation as in Section 4.2. We collect the results in the following corollaries:

Corollary 5.4. ***Perfect plasticity****. Set $\boldsymbol{\zeta}_n = \boldsymbol{A} : \boldsymbol{\varepsilon}_n$, and compute $\Delta\boldsymbol{\varepsilon}_{pn} \in \mathcal{K}^\infty$ and $\boldsymbol{\sigma}_n \in \mathcal{K}^\infty$, and hence $\boldsymbol{\varepsilon}_{np} = \boldsymbol{\varepsilon}_{n-1,p} + \Delta\boldsymbol{\varepsilon}_{np}$, $\boldsymbol{\varepsilon}_{pn} = \boldsymbol{\varepsilon}_n - \boldsymbol{\varepsilon}_{en}$, as the unique solution of*

$$\forall \boldsymbol{\mu} : \quad a^p(\Delta\boldsymbol{\varepsilon}_{pn}, \boldsymbol{\mu}) + \langle \boldsymbol{\sigma}_n, \boldsymbol{\mu} \rangle = \langle \boldsymbol{\zeta}_n, \boldsymbol{\mu} \rangle - a^p(\boldsymbol{\varepsilon}_{p,n-1}, \boldsymbol{\mu}) \tag{99}$$

and

$$\forall \boldsymbol{\tau} \in \mathcal{K} : \quad \langle \Delta\boldsymbol{\varepsilon}_{pn}, \boldsymbol{\tau} - \boldsymbol{\sigma}_n \rangle \le 0. \tag{100}$$

Then this satisfies Eq. (30) and the sequence $\{\boldsymbol{\varepsilon}_{pn}\}, \{\boldsymbol{\sigma}_n\}$ converges as $\Delta t \to 0$ to the solution of Corollary 5.1.

Proof. Eq. (99) is equivalent with

$$\boldsymbol{A} : \Delta\boldsymbol{\varepsilon}_{pn} + \boldsymbol{\sigma}_n = \boldsymbol{\zeta}_n - \boldsymbol{A} : \boldsymbol{\varepsilon}_{p,n-1} = \boldsymbol{A}\boldsymbol{\varepsilon}_n - \boldsymbol{A}\boldsymbol{\varepsilon}_{p,n-1}, \tag{101}$$

and with $\boldsymbol{\sigma}^{trial} = \boldsymbol{A}(\Delta\boldsymbol{\varepsilon}_n + \boldsymbol{\varepsilon}_{e,n-1})$ one has

$$\Delta\boldsymbol{\varepsilon}_{pn} = \boldsymbol{\varepsilon}_n - \boldsymbol{A}^{-1}\boldsymbol{\sigma}_n - \boldsymbol{\varepsilon}_{p,n-1} = \boldsymbol{\varepsilon}_n - \boldsymbol{\varepsilon}_{n-1} + \boldsymbol{\varepsilon}_{e,n-1} - \boldsymbol{A}^{-1}\boldsymbol{\sigma}_n = \boldsymbol{A}^{-1}(\boldsymbol{\sigma}^{trial} - \boldsymbol{\sigma}_n). \tag{102}$$

As stated following Eq. (29), which is itself the same as Eq. (100), this implies Eq. (30). □

For plasticity with general hardening, we define the complementary energy for $\boldsymbol{\Sigma} = (\boldsymbol{\sigma}, \boldsymbol{\chi})$ by

$$a^{g*}(\boldsymbol{\Sigma}_1, \boldsymbol{\Sigma}_2) = a^{p*}(\boldsymbol{\sigma}_2, \boldsymbol{\sigma}_2) + \langle \boldsymbol{\chi}_1, \boldsymbol{H}^{-1}\boldsymbol{\chi}_2 \rangle, \tag{103}$$

where $\boldsymbol{H}^{-1}$ is the inverse of $\boldsymbol{H}$ (see explanation following Eq. (31)). From Corollary 4.4 we then have the analogue of Corollary 5.4:

Corollary 5.5. ***General hardening.*** *Set $\boldsymbol{S}_n = \boldsymbol{S}(t_n)$ from Eq. (93) and compute $\Delta\boldsymbol{E}_{pn}$,$\boldsymbol{E}_{pn} = \boldsymbol{E}_{p,n-1} + \Delta\boldsymbol{E}_{pn}$,and $\boldsymbol{\Sigma}_n$ as the unique solution of*

$$\forall \boldsymbol{M} : a^g(\Delta\boldsymbol{E}_{pn}, \boldsymbol{M}) + \langle \boldsymbol{\Sigma}_n, \boldsymbol{M} \rangle = \langle \boldsymbol{S}_n, \boldsymbol{M} \rangle - a^g(\boldsymbol{E}_{p,n-1}, \boldsymbol{M}) = \langle \boldsymbol{y}_n, \boldsymbol{M} \rangle \tag{104}$$

with $\boldsymbol{y}_n = \boldsymbol{S}_n - a^g(\boldsymbol{E}_{p,n-1}, \cdot))$, and

$$\forall \boldsymbol{T} \in \mathcal{K} : \langle \Delta\boldsymbol{E}_{pn}, \boldsymbol{T} - \boldsymbol{\Sigma}_n \rangle \leq 0. \tag{105}$$

The sequences $\{\boldsymbol{E}_{pn}\}, \{\boldsymbol{\Sigma}_n\}$ converge as $\Delta t \to 0$ to the solution of Corollary 5.2.

5.2 Closest point return algorithm

Here we want to investigate the closest point return algorithm, and from Theorem 4.5 it may be seen that one may formulate the computation with the help of the bilinear, symmetric, and coercive form (the complementary energy)

$$a^{p*}(\boldsymbol{\sigma}, \boldsymbol{\tau}) = \boldsymbol{\sigma} : \boldsymbol{A}^{-1} : \boldsymbol{\tau} \tag{106}$$

in the following way:

Corollary 5.6. ***Perfect plasticity.*** *Set $\boldsymbol{\sigma}^{trial}(= y_n) = \boldsymbol{\zeta}_n - \boldsymbol{A} : \boldsymbol{\varepsilon}_{p,n-1}$. Compute the unique minimizer of*

$$\begin{aligned} \boldsymbol{\sigma}_n &= \arg\min_{\boldsymbol{\sigma}\in\mathcal{K}} \frac{1}{2} a^{p*}(\boldsymbol{\sigma}^{trial} - \boldsymbol{\sigma}, \boldsymbol{\sigma}^{trial} - \boldsymbol{\sigma}) \\ &= \arg\min_{\boldsymbol{\sigma}\in\mathcal{K}} \frac{1}{2} \langle \boldsymbol{\sigma}^{trial} - \boldsymbol{\sigma}, \boldsymbol{A}^{-1} : (\boldsymbol{\sigma}^{trial} - \boldsymbol{\sigma})\rangle \end{aligned} \tag{107}$$

the closest point in $\mathcal{K}$ to $\boldsymbol{\sigma}^{trial}$ in the a^{p*} metric. Set

$$\Delta\boldsymbol{\varepsilon}_{pn} = \boldsymbol{A}^{-1} : (\boldsymbol{\sigma}^{trial} - \boldsymbol{\sigma}_n), \tag{108}$$

then $\Delta\boldsymbol{\varepsilon}_{pn}$ and $\boldsymbol{\sigma}_n$ solve the problem in Corrolary 5.4. With $\boldsymbol{\varepsilon}_{pn} = \boldsymbol{\varepsilon}_{p,n-1} + \Delta\boldsymbol{\varepsilon}_{p,n-1}$ the sequence $\{\boldsymbol{\varepsilon}_{pn}\}, \{\boldsymbol{\sigma}_n\}$ converges as $\Delta t \to 0$ to the solution of Corrolary 5.1.

We see that this is exactly what is stated in Eq. (29) and Eq. (30).

The closest point projection algorithm in case of general hardening is then analogous to solution to Corollary 5.6:

Corollary 5.7. ***General hardening:*** *Set $\boldsymbol{\Sigma}^{trial} = \boldsymbol{y}_n = \boldsymbol{S}_n - a^g(\boldsymbol{E}_{p,n-1}, \cdot)$. Compute the unique minimizer of:*

$$\boldsymbol{\Sigma}_n = \arg\min_{\boldsymbol{\Sigma}\in\mathcal{K}} \frac{1}{2} a^{g*}(\boldsymbol{\Sigma}^{trial} - \boldsymbol{\Sigma}, \boldsymbol{\Sigma}^{trial} - \boldsymbol{\Sigma}) \tag{109}$$

and $\Delta\boldsymbol{E}_{pn}$ as unique solution of

$$\forall \boldsymbol{M} : \ a^g(\Delta\boldsymbol{E}_{pn}, \boldsymbol{M}) = \langle \boldsymbol{y}_n - \boldsymbol{\Sigma}_n, \boldsymbol{M}\rangle. \tag{110}$$

Then $\Delta\boldsymbol{E}_{pn}$ and $\boldsymbol{\Sigma}_n$ solve the problem in Corollary 5.5 and $\Delta\boldsymbol{E}_{pn} \in \mathcal{K}^\infty$ satisfies

$$\forall \boldsymbol{T} \in \mathcal{K} : \langle \Delta\boldsymbol{E}_{pn}, \boldsymbol{T} - \boldsymbol{\Sigma}\rangle \leq 0. \tag{111}$$

Set $\boldsymbol{E}_{pn} = \boldsymbol{E}_{p,n-1} + \Delta\boldsymbol{E}_{pn}$. The sequences $\{\boldsymbol{E}_{pn}\}, \{\boldsymbol{\Sigma}_n\}$ converge as $\Delta t \to 0$ to the solution of the problem in Corollary 5.2.

The special case of combined isotropic and kinematic hardening is covered by Corollaries 5.5 and 5.7 with the identifications at the end of Section 2.4.

Corollary 5.8. *Just as the time continuous combined hardening problem in Corollary 5.3 is a special case of the general hardening problem Corollary 5.2, so is the time discrete version. Therefore the closest point algorithm from Corollary 4.5 and Corollary 5.7 also yields converegent sequences $\{\boldsymbol{\varepsilon}_{pn}\}, \{\boldsymbol{\sigma}_n\}, \{\nu_n\}$ as $\Delta t \to 0$.*

Let us remark that Theorem 4.5 and Corollaries 5.6, and 5.7 can lead to a completely dual formulation [14]. But the computations are usually performed according to the mixed formulation Theorem 4.2 (ABS-M) on a global level, whereas upon discretisation at each Gauss-point the local computation is usually done according to the dual formulation of Theorem 4.5 and Corrolaries 5.6, and 5.7 [32, 16].

6 Deterministic plasticity

In order to formulate and understand the stochastic elastoplastic problem, we first need to recall the formulation of the deterministic counterpart of the problem. Instead of one material point as in Section 2, the equations and inequalities are now posed on the whole computational domain $\mathcal{G}$. This further allow us in Section 7 to describe the stochastic problem as a parametrisation of the corresponding deterministic problem.

6.1 Functional Spaces

The following section recalls the function spaces [14] that are relevant to the problem of plasticity with mixed linear hardening. Let us introduce the primal variable $\boldsymbol{w} := (\boldsymbol{u}, \boldsymbol{E}_p) \in \mathcal{Z}$, which exists in the so called "strain" space $\mathcal{Z} := \mathcal{U} \times \mathcal{P}$, together with its dual variable $\boldsymbol{w}^* := (\boldsymbol{f}, \boldsymbol{\Sigma}) \in \mathcal{Z}^* := \mathcal{F} \times \mathcal{Y}$ defined in the "stress" space $\mathcal{Z}^*$ such that the duality pairing is interpreted as $\langle \boldsymbol{w}^*, \boldsymbol{w} \rangle_{\mathcal{Z}^* \times \mathcal{Z}} := \langle \boldsymbol{f}, \boldsymbol{u} \rangle_{\mathcal{F} \times \mathcal{U}} + \langle \boldsymbol{\Sigma}, \boldsymbol{E}_p \rangle_{\mathcal{Y} \times \mathcal{P}}$. Following this, let us specify the spaces of variables under consideration together with the corresponding duality pairings:

- the space of displacement and forces:

$$
\begin{aligned}
&\text{a) displacement: } \mathcal{U} := \{\boldsymbol{u} \in H^1(\mathcal{G}) \mid \boldsymbol{u} = \boldsymbol{0} \text{ on } \Gamma_D\} \\
&\text{b) force: } \mathcal{F} := \mathcal{U}^* \\
&\text{c) } \langle \boldsymbol{f}, \boldsymbol{u} \rangle_{\mathcal{F} \times \mathcal{U}} := \int_{\mathcal{G}} \boldsymbol{f}(x) \cdot \boldsymbol{u}(x)\, \mathrm{d}x + \int_{\Gamma_N} \boldsymbol{g}(x) \cdot \boldsymbol{u}(x)\, \mathrm{d}s,
\end{aligned}
\tag{112}
$$

- the space of deformation and stress:

$$
\begin{aligned}
&\text{a) deformation: } \mathcal{E} := \{\boldsymbol{\varepsilon} \mid \boldsymbol{\varepsilon} \in L^2(\mathcal{G}, \mathrm{Sym}(\mathbb{R})^d)\} \\
&\text{b) stress: } \mathcal{R} := \{\boldsymbol{\sigma} \colon \boldsymbol{\sigma} \in L^2(\mathcal{G}, \mathrm{Sym}(\mathbb{R}^d)\} \\
&\text{c) } \langle \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \rangle_{\mathcal{R} \times \mathcal{E}} := \int_{\mathcal{G}} \boldsymbol{\sigma}(x) : \boldsymbol{\varepsilon}(x)\, \mathrm{d}x
\end{aligned}
\tag{113}
$$

- and the space of hardening internal variables and corresponding conjugate forces:

$$\begin{aligned} &\text{a) int. var.: } \mathcal{Q} := \{\boldsymbol{\eta} \mid \boldsymbol{\eta} = (\boldsymbol{\varepsilon}_p, \nu) \in L_2(\mathcal{G}, \mathrm{Sym}(\mathbb{R})^d \times \mathbb{R})\}, \\ &\text{b) conj. f.: } \mathcal{C} := \{\boldsymbol{\chi} \mid \boldsymbol{\chi} = (\boldsymbol{\varsigma}, \zeta) \in L_2(\mathcal{G}, \mathrm{Sym}(\mathbb{R}^d) \times \mathbb{R}\}, \\ &\text{c) } \langle \boldsymbol{\chi}, \boldsymbol{\eta} \rangle_{\mathcal{C}\times\mathcal{Q}} := \int_{\mathcal{G}} \boldsymbol{\varsigma}(x) : \boldsymbol{\varepsilon}_p(x)\ \mathrm{d}x + \int_{\mathcal{G}} \zeta(x) \cdot \nu(x)\ \mathrm{d}x. \end{aligned} \tag{114}$$

Shortly in terms of generalised stress and strain one has:

$$\begin{aligned} &\text{a) } \boldsymbol{\Sigma} := (\boldsymbol{\sigma}, \boldsymbol{\chi}) = (\boldsymbol{\sigma}, \boldsymbol{\varsigma}, \zeta) \in \mathcal{Y} := \mathcal{R} \times \mathcal{C}, \\ &\text{b) } \boldsymbol{E}_p := (\boldsymbol{\varepsilon}_p, \boldsymbol{\eta}) = (\boldsymbol{\varepsilon}_p, \boldsymbol{\varepsilon}_p, \nu) \in \mathcal{P} := \mathcal{E} \times \mathcal{Q}, \\ &\text{c) } \langle \boldsymbol{\Sigma}, \boldsymbol{E}_p \rangle_{\mathcal{Y}\times\mathcal{P}} := \langle \boldsymbol{\sigma}, \boldsymbol{\varepsilon}_p \rangle_{\mathcal{R}\times\mathcal{E}} + \langle \boldsymbol{\chi}, \boldsymbol{\eta} \rangle_{\mathcal{C}\times\mathcal{Q}}. \end{aligned} \tag{115}$$

6.2 Problem formulation

Let $\boldsymbol{w} = (\boldsymbol{u}, \boldsymbol{E}_p) = (\boldsymbol{u}, (\boldsymbol{\varepsilon}_p, \boldsymbol{\eta})) = (\boldsymbol{u}, (\boldsymbol{\varepsilon}_p, \boldsymbol{\varepsilon}_p, \nu)) \in \mathcal{Z}$, then the total strain due to a displacement $\boldsymbol{u}$ is given by Eq. (2). As we consider linear mixed hardening (see Section 2.4), the corresponding bilinear form is given by

$$\begin{aligned} a^d(\boldsymbol{w}_1, \boldsymbol{w}_2) \quad = \quad & \int_{\mathcal{G}} (\boldsymbol{\varepsilon}_x(\boldsymbol{u}_1) - \boldsymbol{\varepsilon}^1_{px}) : \boldsymbol{A}(x) : (\boldsymbol{\varepsilon}_x(\boldsymbol{u}_2) - \boldsymbol{\varepsilon}^2_{px})\ \mathrm{d}x + \\ & \int_{\mathcal{G}} H_{iso}(x) \nu_{1x} \nu_{x2}\ \mathrm{d}x + \boldsymbol{H}_{kin}(x) \boldsymbol{\varepsilon}^1_{px} : \boldsymbol{\varepsilon}^2_{px}\ \mathrm{d}x, \end{aligned} \tag{116}$$

and has all the required properties [14] for Proposition 4.5 and hence Theorem 4.2 to hold. As $\boldsymbol{H}_{kin}(x)$ and $H_{iso}(x)$ are positive definite, the bilinear form a^d is elliptic on $\mathcal{Z} = \mathcal{U} \times \mathcal{P}$ and further symmetric and bounded.

The linear functional is given as:

$$\langle \boldsymbol{f}, \boldsymbol{z} \rangle = \int_{\mathcal{G}} \boldsymbol{f} \cdot \boldsymbol{v}\ \mathrm{d}x \tag{117}$$

and the dissipation functional as:

$$j(\boldsymbol{z}) = \int_{\mathcal{G}} j_d(\boldsymbol{\epsilon}, \mu)\ \mathrm{d}x, \tag{118}$$

where $j_d(\boldsymbol{\epsilon}, \mu) = \mathrm{const}\ |\boldsymbol{\epsilon}|$ if $\boldsymbol{\epsilon} \le \mu$, otherwise $j_d = \infty$. Here the isotropic internal variable μ is chosen as the equivalent plastic strain. This leads to the definition of the convex domain via a yield function ϕ as:

$$\mathcal{K} = \{\boldsymbol{\Sigma} \in \mathcal{Y} :\ \phi(\boldsymbol{\Sigma}) \le 0 \text{ a.e. on } \mathcal{G}\}. \tag{119}$$

The setting just introduced describes the deterministic plasticity problem in the primal formulation:

Proposition 6.1. *With the same notation and assumptions as before the following primal deterministic plasticity problem is well posed:*

Problem DP-P: *given a function $\tilde{\boldsymbol{f}} \in H^1(\mathcal{T}, \mathcal{F})$ with $\tilde{\boldsymbol{f}}(0) = 0$, set $\boldsymbol{f} = [\tilde{\boldsymbol{f}}, 0] \in H^1(\mathcal{T}, \mathcal{F} \times \mathcal{Y})$. Then there exists a unique function $\boldsymbol{w} = (\boldsymbol{u}, (\boldsymbol{\varepsilon}_p, \nu)) \in H^1(\mathcal{T}, \mathcal{Z})$ with $\boldsymbol{w}(0) = 0$,$\dot{\boldsymbol{w}}(t) \in \mathcal{K}^\infty$ which solves a.e. in $\mathcal{T}$*

$$a^d(\boldsymbol{w}(t), \boldsymbol{z} - \dot{\boldsymbol{w}}(t)) + j(\boldsymbol{z}) - j(\dot{\boldsymbol{w}}) \geq \langle \boldsymbol{f}, \boldsymbol{z} - \dot{\boldsymbol{w}}(t) \rangle \tag{120}$$

for all $\boldsymbol{z} = (\boldsymbol{v}, (\boldsymbol{\mu}, \upsilon)) \in \mathcal{Z} = \mathcal{U} \times \mathcal{P}$. If in addition $\tilde{\boldsymbol{f}}_1, \tilde{\boldsymbol{f}}_2 \in H^1(\mathcal{T}, \mathcal{F})$ are two different loadings, and $\boldsymbol{w}_1, \boldsymbol{w}_2 \in H^1(\mathcal{T}, \mathcal{Z})$ are the corresponding solutions, then:

$$\|\boldsymbol{w}_1 - \boldsymbol{w}_2\|_{L_\infty} \leq c \|\dot{\tilde{\boldsymbol{f}}}_1 - \dot{\tilde{\boldsymbol{f}}}_2\|_{L_1}. \tag{121}$$

Proof. This is esentially a re-statement of Proposition 4.1. □

Let us recall from [14] that such a result is not attainable for general hardening or perfect plasticity. In those cases the bilinear form is not coercive, and uniqueness at least is lost.

To obtain a formulation, which corresponds to the dicretisation most often used in practice, we give:

Theorem 6.2. *With the same notation and assumptions as before the following mixed deterministic plasticity problem with combined hardening is well posed:*

Problem DP-M: *there are functions $\boldsymbol{w} = (\boldsymbol{u}, \boldsymbol{E}_p) \in H^1(\mathcal{T}, \mathcal{Z})$ with $\boldsymbol{w}(0) = 0$ and $\boldsymbol{w}^* \in H^1(\mathcal{T}, \mathcal{Z}^*), \mathcal{Z}^* = \mathcal{F} \times \mathcal{Y}, \boldsymbol{w}^*(0) = 0$ such that a.e. in $\mathcal{T}, \dot{w} \in \mathcal{K}^\infty = \partial j(0), \forall \boldsymbol{z} = (\boldsymbol{v}, (\boldsymbol{\mu}, \upsilon)) \in \mathcal{Z}$:*

$$a^d(\boldsymbol{w}(t), \boldsymbol{z}) + \langle \dot{\boldsymbol{w}}(t), \boldsymbol{z} \rangle = \langle \boldsymbol{f}, \boldsymbol{z} \rangle \tag{122}$$

and

$$\forall \boldsymbol{z}^* \in \mathcal{K} : \quad \langle \dot{\boldsymbol{w}}, \boldsymbol{z}^* - \boldsymbol{w}^* \rangle \leq 0. \tag{123}$$

If in addition $\boldsymbol{f}_1, \boldsymbol{f}_2 \in H^1(\mathcal{T}, \mathcal{F})$ are two different loadings, and $\boldsymbol{w}_1, \boldsymbol{w}_2 \in H^1(\mathcal{T}, \mathcal{Z})$ and $\boldsymbol{w}_1^*, \boldsymbol{w}_2^* \in H^1(\mathcal{T}, \mathcal{Z}^*)$ the corresponding solutions, then:

$$\|\boldsymbol{w}_1 - \boldsymbol{w}_2\|_{L_\infty} \leq c \|\dot{\boldsymbol{f}}_1 - \dot{\boldsymbol{f}}_2\|_{L_1} \tag{124}$$

and

$$\|\boldsymbol{w}_1^* - \boldsymbol{w}_2^*\|_{L_\infty} \leq c^* \|\dot{\boldsymbol{f}}_1 - \dot{\boldsymbol{f}}_2\|_{L_1} \tag{125}$$

Proof. This is a direct re-statement of Theorem 4.2. □

6.3 Time discretisation

Corresponding to the primal and mixed formulation (Proposition 6.1 and Theorem 6.2) there are two statements on time discretisations:

Proposition 6.3. ***Problem Discrete DP-DP*** *the incerement* $\Delta \boldsymbol{w}_n = (\Delta \boldsymbol{u}, \Delta \boldsymbol{\varepsilon}_p, \Delta \boldsymbol{\eta}) = (\Delta \boldsymbol{u}, (\Delta \boldsymbol{\varepsilon}_p, \Delta \boldsymbol{\varepsilon}_p, \Delta \nu)) \in \mathcal{K}^\infty$ *satisfies for all* $\boldsymbol{z} = (\boldsymbol{v}, (\boldsymbol{\mu}, \upsilon)) \in \mathcal{Z} = \mathcal{U} \times \mathcal{P}$*:*

$$a^d(\Delta \boldsymbol{w}_n, \boldsymbol{z} - \boldsymbol{w}_n + j(\boldsymbol{z}(-j(\Delta \boldsymbol{w}_n) \geq \\ \langle \boldsymbol{f}(t_n), \boldsymbol{z} - \Delta \boldsymbol{w}_n \rangle - a^d(\boldsymbol{w}_{n-1}, \boldsymbol{z} - \Delta \boldsymbol{w}_n) \tag{126}$$

such that the approximate solution $(\boldsymbol{w}_0, \boldsymbol{w}_1, .., \boldsymbol{w}_n, ...)$ *converges to the solution* $\boldsymbol{w}(t)$ *of problem DP–P in Proposition 6.1 as* $\Delta t \to 0$.

Proof. This is a re-statement of Proposition 4.3. □

For the mixed formulation, we obtain a new result:

Theorem 6.4. *Let* $\Delta \boldsymbol{w}_n \in \mathcal{K}^\infty \subset \mathcal{Z}$ *be unique solution in Proposition 6.3 such that for all* t_n *there exists* $\boldsymbol{w}_n^* = (\boldsymbol{\sigma}_n, \boldsymbol{\chi}_n) \in \mathcal{K} \subset \mathcal{Z}^*$ *satisfying:*

$$a^d(\Delta \boldsymbol{w}_n, \boldsymbol{z}) + \langle \boldsymbol{w}_n^*, \boldsymbol{z} \rangle = \langle \boldsymbol{y}_n, \boldsymbol{z} \rangle \tag{127}$$

for all $\boldsymbol{z} = (\boldsymbol{v}, (\boldsymbol{\mu}, \upsilon)) \in \mathcal{Z} = \mathcal{U} \times \mathcal{P}$ *and*

$$\langle \Delta \boldsymbol{w}_n, \boldsymbol{z}^* - \boldsymbol{w}_n^* \rangle \leq 0 \tag{128}$$

for all $\boldsymbol{z}^* = (\boldsymbol{\tau}, \boldsymbol{\xi}, \nu) \in \mathcal{K}$*. The approximate solutions* $\{\boldsymbol{w}_n\}, \{\boldsymbol{w}_n^*\}$ *converge as* $\Delta t \to 0$ *to the solutions* $\boldsymbol{w}(t)$ *and* $\boldsymbol{w}^*(t)$ *of problem DP-M in Theorem 6.2.*

Proof. This is a re-statement of Theorem 4.4 □

The computation may be performed by the closest point-return algorithm, as showed to be a re–statement of Theorem 4.5, and thus is not repeated here.

7 Stochastic problem

The theory presented in previous section is restricted to the case when one has the precise knowledge about material parameters, which however in reality can not be known at every point of domain. Thus, the problem has to be extended to a more general one which takes into consideration existing uncertanties. Let us introduce the consititutive tensor $\boldsymbol{A}(x,\omega)$, the hardening tensor $\boldsymbol{H}(x,\omega)$ and the yield stress $\sigma_y(x,\omega)$ as uncertain positive definite parameters, here modelled as lognormal random fields according to the maximum entropy principle. These random fields are defined over the probability space $(\Omega,\mathcal{B},\mathbb{P})$, where Ω is the space of elementary events, $\mathcal{B}\subset 2^{\Omega}$ a σ-algebra of subsets of Ω, and $\mathbb{P}$ a probability measure. In addition, one may assume that the external loading $\boldsymbol{f}(x,\omega)$ is uncertain as well. The mentioned uncertainties make the analysis more complex but however similar to the one given in previous sections.

7.1 Functional spaces

In contrast to the deterministic problem which is fully defined in the space described in Section 6, the stochastic problem requires the introduction of the linear space $\mathscr{V}=L_2(\Omega,\mathcal{B},\mathbb{P};\mathcal{V})$, where $\mathcal{V}$ is a Hilbert space of consideration such that for $u\in\mathcal{V}$ and $\omega\in\Omega$:

$$u(\cdot,\omega)\in\mathcal{V}, \tag{129}$$

and for $x\in\mathcal{G}$:

$$u(x,\cdot)\in L_2(\Omega,\mathcal{B},\mathbb{P}). \tag{130}$$

According to this the variables u live in a space obtained as a tensor product of the corresponding deterministic space $\mathcal{V}$ and the stochastic space (S). The choice of (S) and hence the stochastic regularity of the solution depends on the stochastic regularity of the right hand side and parameters. Here, for the sake of simplicity we assume $(S)=L_2(\Omega)$. Thus, one has:

$$\mathscr{V}\simeq\mathcal{V}\otimes(S), \tag{131}$$

which is the Hilbert space induced by inner product $\langle\!\langle\boldsymbol{u}|\boldsymbol{v}\rangle\!\rangle_{\mathscr{V}}=\mathbb{E}(\langle\boldsymbol{u}|\boldsymbol{v}\rangle_{\mathcal{V}})$ and duality pairing:

$$\langle\!\langle\boldsymbol{v},\boldsymbol{u}\rangle\!\rangle_{\mathscr{V}}:=\mathbb{E}(\langle\boldsymbol{v},\boldsymbol{u}\rangle_{\mathcal{V}}), \tag{132}$$

where $\mathbb{E}(\cdot):=\int_{\Omega}(\cdot)\mathbb{P}(\,\mathrm{d}\omega)$ is the mathematical expectation taken with respect to the probability measure $\mathbb{P}$.

Following previous definitions, we may formulate the spaces describing the stochastic linear mixed hardening plasticity as:

- the space of random displacement and forces:

$$\begin{aligned}
&\text{a) displacement: } \mathscr{U}:=\mathcal{U}\otimes(S)\\
&\text{b) force: } \mathscr{F}:=\mathcal{F}\otimes(S)=\mathcal{U}^*\otimes(S)\\
&\text{c) } \langle\!\langle\boldsymbol{f},\boldsymbol{u}\rangle\!\rangle_{\mathscr{F}\otimes\mathscr{U}}:=\mathbb{E}\left(\langle\boldsymbol{f},\boldsymbol{u}\rangle_{\mathcal{F}\times\mathcal{U}}\right),
\end{aligned} \tag{133}$$

- the space of random deformation and stress:

$$\begin{aligned} &\text{a) deformation: } \mathscr{E} := \mathcal{E} \otimes (S) \\ &\text{b) stress: } \mathscr{R} = \mathcal{R} \otimes (S) \\ &\text{c) } \langle\!\langle \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \rangle\!\rangle_{\mathscr{R} \otimes \mathscr{E}} := \mathbb{E}\left(\langle \boldsymbol{\varepsilon}, \boldsymbol{\sigma} \rangle_{\mathcal{R} \times \mathcal{E}} \right) \end{aligned} \tag{134}$$

- the space of random internal hardening variables and correspondng conjugate forces:

$$\begin{aligned} &\text{a) internal variables: } \mathscr{Q} = \mathcal{Q} \otimes (S), \\ &\text{b) conjugate forces: } \mathscr{C} = \mathcal{C} \otimes (S), \\ &\text{c) } \langle \boldsymbol{\chi}, \boldsymbol{\eta} \rangle_{\mathscr{C} \times \mathscr{Q}} := \mathbb{E}(\langle \boldsymbol{\chi}, \boldsymbol{\eta} \rangle_{\mathcal{C} \times \mathcal{Q}}) \end{aligned} \tag{135}$$

- the spaces of random generalised plastic strain and stress:

$$\begin{aligned} &\text{a) generalised plastic strain: } \boldsymbol{E}_p := (\boldsymbol{\varepsilon}_p, \boldsymbol{\eta}) \in \mathscr{P} := \mathcal{P} \otimes (S), \\ &\text{b) generalised stress: } \boldsymbol{\Sigma} := (\boldsymbol{\sigma}, \boldsymbol{\chi}) \in \mathscr{Y} := \mathcal{Y} \otimes (S), \\ &\text{c) } \langle\!\langle \boldsymbol{\Sigma}, \boldsymbol{E}_p \rangle\!\rangle_{\mathscr{Y} \otimes \mathscr{P}} := \mathbb{E}\left(\langle \boldsymbol{\Sigma}, \boldsymbol{E}_p \rangle_{\mathcal{Y} \times \mathcal{P}} \right). \end{aligned} \tag{136}$$

Moreover, the primal variable $\boldsymbol{w} := (\boldsymbol{u}, \boldsymbol{E}_p)$ belongs to a space $\mathscr{Z} = \mathscr{U} \times \mathscr{P}$, with dual space $\mathscr{Z}^* = \mathscr{F} \times \mathscr{Y}$. With respect to this, the description of elastoplastic behaviour including uncertainty is given in the same framework as in Section 2 with the only difference that the constitutive and evolution laws must hold almost surely. In addition, the differentiation is done in a weak sense such that for a single tensor product $\boldsymbol{u}_1(x)\boldsymbol{u}_2(\omega) \in \mathscr{U} := \mathcal{U} \otimes (S)$ one has $\nabla_S : \boldsymbol{u}_1(x)\boldsymbol{u}_2(\omega) \mapsto (\nabla_S \boldsymbol{u}_1(x))\boldsymbol{u}_2(\omega)$. By linearity and continuity this can be extended to a linear bounded symmetric operator: $\boldsymbol{\nabla}_S = (\nabla_S \otimes \boldsymbol{I}) : \mathcal{U} \otimes (S) \to \mathscr{E}$.

7.2 Problem formulation

Let us define the elastic domain as a closed convex set

$$\mathscr{Y} \supset \tilde{\mathscr{K}} = \{(\boldsymbol{\sigma}, \boldsymbol{\chi}) \in \mathscr{R} \times \mathscr{C} \mid \phi_{\mathscr{K}}(x, \omega, \boldsymbol{\sigma}_x, \boldsymbol{\chi}) \leq 0, \mathbb{P} - \text{a.s.}\} \tag{137}$$

described by a yield function $\phi_{\mathscr{K}}(x, \omega, \boldsymbol{\sigma}_x, \boldsymbol{\chi})$, and its barier cone $\tilde{\mathscr{K}}^\infty \subseteq \mathscr{P}$. Then the set of admissible stress states becomes $\mathscr{F} \times \mathscr{Y} \supset \mathscr{K} = \{(\boldsymbol{f}, \boldsymbol{\sigma}, \boldsymbol{\chi}) \mid (\boldsymbol{\sigma}, \boldsymbol{\chi}) \in \tilde{\mathscr{K}}\}$ and the barier cone $\mathscr{K}^\infty = \{0\} \times \tilde{\mathscr{K}}^\infty$, which allow us to extend the mathematical theory given in previous sections to the more general one including uncertanties of specified parameters or the right hand side.

Similarly to the determinitistic case, we consider the problem of linear mixed hardening plasticity described by a bilinear form a^s as a rephrasing of those given by Eq. (116):

$$\begin{aligned} a^s(\boldsymbol{w}_1, \boldsymbol{w}_2) &= \int_\Omega \int_\mathcal{G} (\boldsymbol{\varepsilon}_x(\boldsymbol{u}_1) - \boldsymbol{\varepsilon}^1_{px}) : \boldsymbol{A} : (\boldsymbol{\varepsilon}_x(\boldsymbol{u}_2) - \boldsymbol{\varepsilon}^2_{px}) \,\mathrm{dx}\, \mathbb{P}(\mathrm{d}\omega) \\ &\quad + \int_\Omega \int_\mathcal{G} H_{iso} \nu^1_x \nu^2_x \,\mathrm{dx}\, \mathbb{P}(\mathrm{d}\omega) \\ &+ \int_\Omega \int_\mathcal{G} H_{kin} \boldsymbol{\varepsilon}^1_{px} : \boldsymbol{\varepsilon}^2_{px} \,\mathrm{dx}\, \mathbb{P}(\mathrm{d}\omega). \end{aligned} \tag{138}$$

In order to make sure that we may apply Theorem 4.4 we strive to have similar overall properties of the system as in the deterministic case (Theorem 6.2). For this to hold, it is necessary that the operator defined by the bilinear form a^s is continuous and continuously invertible, i.e. we require that both the constitutive tensor $\boldsymbol{A}(x,\omega) \in L_\infty(\mathcal{G} \times \Omega)$ and compliance tensor $\boldsymbol{A}^{-1}(x,\omega) \in L_\infty(\mathcal{G} \times \Omega)$:

$$\boldsymbol{A}_+ \geq \|\boldsymbol{A}(x,\omega)\| \geq \boldsymbol{A}_- > 0 \text{ a.e. and a.s.} \tag{139}$$

Similar assumptions are made for the hardening tensor $\boldsymbol{H}$ and its inverse $\boldsymbol{H}^{-1}$, i.e.

$$\boldsymbol{H}_+ \geq \|\boldsymbol{H}(x,\omega)\| \geq \boldsymbol{H}_- > 0 \text{ a.e. and a.s.} \tag{140}$$

Taking the assumptions Eq. (139) and Eq. (140), the bilinear form a^s is $\mathscr{Z} = \mathscr{U} \times \mathscr{P}$ elliptic and it is obviously symmetric and bounded.

In addition, let us for $\boldsymbol{z} \in \mathscr{Z}$ define:

$$\langle\!\langle \boldsymbol{f}, \boldsymbol{z} \rangle\!\rangle = \int_\Omega \int_\mathcal{G} \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}x\, \mathbb{P}(\mathrm{d}\omega) \tag{141}$$

where $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ is the duality paring between $(\mathcal{Z} \otimes (S))^* = \mathcal{Z}^* \otimes (S)^*$ and $\mathcal{Z} \otimes (S)$. Similarly, following the defintion of j_d in Eq. (118) we may write:

$$j(\boldsymbol{z}) = \int_\Omega \int_\mathcal{G} j_d(\boldsymbol{\varepsilon}_p, \nu) \,\mathrm{d}x\, \mathbb{P}(\mathrm{d}\omega), \tag{142}$$

which together with previous defintions allow us to recast the primal formulation of the elastoplastic problem to the analogue of Proposition 6.1:

Theorem 7.1. ***Problem SP-P:*** *given a function $\tilde{\boldsymbol{f}} \in H^1(\mathcal{T}, \mathscr{F})$ with $\tilde{\boldsymbol{f}}(0) = 0$, set $\boldsymbol{f} = [\tilde{\boldsymbol{f}}, 0] \in H^1(\mathcal{T}, \mathscr{F} \times \mathscr{Y})$. Then there exists a unique function $\boldsymbol{w} = (\boldsymbol{u}, \boldsymbol{E}_p) \in H^1(\mathcal{T}, \mathscr{Z})$ with $\boldsymbol{w}(0) = 0$ and $\dot{\boldsymbol{w}}(t) \in \mathscr{K}^\infty$ which solves a.s. in Ω, a.e. in $\mathcal{T}$:*

$$a^s(\boldsymbol{w}(t), \boldsymbol{z} - \dot{\boldsymbol{w}}(t)) + j(\boldsymbol{z}) - j(\dot{\boldsymbol{w}}) \geq \langle\!\langle \boldsymbol{f}, \boldsymbol{z} - \dot{\boldsymbol{w}}(t) \rangle\!\rangle \tag{143}$$

for all $\boldsymbol{z} = (\boldsymbol{v}, (\boldsymbol{\mu}, \upsilon)) \in \mathscr{Z}$. *If in addition* $\tilde{\boldsymbol{f}}_1, \tilde{\boldsymbol{f}}_2 \in H^1(\mathcal{T}, \mathscr{F})$ *are two different loadings,* $\boldsymbol{w}_1, \boldsymbol{w}_2 \in H^1(\mathcal{T}, \mathscr{Z})$ *are the corresponding solutions, then:*

$$\|\boldsymbol{w}_1 - \boldsymbol{w}_2\|_{L_\infty(\mathcal{T},\mathcal{Z}} \leq c\|\dot{\boldsymbol{f}}_1 - \dot{\boldsymbol{f}}_2\|_{L_1(\mathcal{T},\mathcal{Z})}. \tag{144}$$

Proof. Follows from Proposition 6.1. □

Similarly, for the mixed stochastic plasticity problem with combined hardening one has the analogue of Theorem 6.2:

Theorem 7.2. ***Problem SM-P****: there are functions* $\boldsymbol{w} = (\boldsymbol{u}, \boldsymbol{\eta}) \in H^1(\mathcal{T}, \mathscr{Z})$ *with* $\boldsymbol{w}(0) = 0$, $\boldsymbol{w}^* \in H^1(\mathcal{T}, \mathscr{Z}^*)$ *and* $\boldsymbol{w}^*(0) = 0, \dot{w} \in \mathscr{K}^\infty$ *such that a.s. in* Ω, *a.e. in* $\mathcal{T}$*:*

$$a^s(\boldsymbol{w}(t), \boldsymbol{z}) + \langle\!\langle \dot{\boldsymbol{w}}(t), \boldsymbol{z} \rangle\!\rangle = \langle\!\langle \boldsymbol{f}, \boldsymbol{z} \rangle\!\rangle \tag{145}$$

for all $\boldsymbol{z} = (\boldsymbol{v}, (\boldsymbol{\mu}, \upsilon)) \in \mathscr{Z}$ *and*

$$\langle\!\langle \dot{\boldsymbol{w}}, \boldsymbol{z}^* - \boldsymbol{w}^* \rangle\!\rangle \leq 0 \tag{146}$$

for all $\boldsymbol{z}^* \in \mathscr{K} \subset \mathcal{Z}^*$. *If in addition* $\tilde{\boldsymbol{f}}_1, \tilde{\boldsymbol{f}}_2 \in H^1(\mathcal{T}, \mathscr{F})$ *are two different loadings, and* $\boldsymbol{w}_1, \boldsymbol{w}_2 \in H^1(\mathcal{T}, \mathscr{Z})$ *and* $\boldsymbol{w}_1^*, \boldsymbol{w}_2^* \in H^1(\mathcal{T}, \mathscr{Z}^*)$ *the corresponding solutions, then:*

$$\|\boldsymbol{w}_1 - \boldsymbol{w}_2\|_{L_\infty(\mathcal{T},\mathcal{Z})} \leq c\|\dot{\boldsymbol{f}}_1 - \dot{\boldsymbol{f}}_2\|_{L_1(\mathcal{T},\mathcal{Z})} \tag{147}$$

and

$$\|\boldsymbol{w}_1^* - \boldsymbol{w}_2^*\|_{L_\infty(\mathcal{T},\mathcal{Z})} \leq c^*\|\dot{\boldsymbol{f}}_1 - \dot{\boldsymbol{f}}_2\|_{L_1(\mathcal{T},\mathcal{Z})}. \tag{148}$$

Proof. Follows from Theorem 4.2. □

7.3 Time discretisation

The time discretisation is performed in the same way as in the deterministic case, and one has the analogue of Theorem 6.3 for the time discretisation of the problem in Theorem 7.1:

Theorem 7.3. ***Problem Discrete SP-DP*** *the increment* $\Delta\boldsymbol{w}_n = (\Delta\boldsymbol{u}, \Delta\boldsymbol{E}_p) \in \mathscr{K}^\infty$ *satisfies for all* $\boldsymbol{z} = (\boldsymbol{v}, (\boldsymbol{\mu}, \nu^*)) \in \mathcal{Z}$*:*

$$\begin{aligned} a^s(\Delta\boldsymbol{w}_n, \boldsymbol{z} - \boldsymbol{w}_n) + j(\boldsymbol{z}) - j(\Delta\boldsymbol{w}_n) \geq \\ \langle\!\langle \boldsymbol{f}(t_n), \boldsymbol{z} - \Delta\boldsymbol{w}_n \rangle\!\rangle - a^s(\boldsymbol{w}_{n-1}, \boldsymbol{z} - \Delta\boldsymbol{w}_n) \end{aligned} \tag{149}$$

such that approximate solution $(\boldsymbol{w}_0, \boldsymbol{w}_1, .., \boldsymbol{w}_n, ...)$ *converges to the solution* $\boldsymbol{w}(t)$ *of problem SP-P in Theorem 7.1 as* $\Delta t \to 0$.

Similarly, we get a result for the mixed formulation $SM-P$ from Theorem 7.2 and Theorem 6.2, the analogue of Theorem 6.4:

Theorem 7.4. *Let $\Delta \boldsymbol{w}_n \in \mathscr{K}^\infty \subset \mathscr{Z}$ be the unique solution such that for each t_n exists $\boldsymbol{w}_n^* = (\boldsymbol{\sigma}_n, \boldsymbol{\chi}_n) \in \mathscr{K} \subset \mathscr{Z}^*$ satisfying:*

$$a^s(\Delta \boldsymbol{w}_n, \boldsymbol{z}) + \langle\langle \boldsymbol{w}_n^*, \boldsymbol{z} \rangle\rangle = \langle\langle \boldsymbol{y}_n, \boldsymbol{z} \rangle\rangle \tag{150}$$

for all $\boldsymbol{z} = (\boldsymbol{v}, (\boldsymbol{\mu}, \upsilon)) \in \mathscr{Z} = \mathscr{U} \times \mathscr{P}$: and

$$\forall \boldsymbol{z}^* = (\boldsymbol{\tau}, \boldsymbol{\xi}, \iota) \in \mathscr{K} : \; \langle\langle \Delta \boldsymbol{w}_n, \boldsymbol{z}^* - \boldsymbol{w}_n^* \rangle\rangle \leq 0. \tag{151}$$

The approximate solutions $\{\boldsymbol{w}_n\}, \{\boldsymbol{w}_n^*\}$ converge as $\Delta t \to 0$ to the solutions $\boldsymbol{w}(t)$ and $\boldsymbol{w}^*(t)$ of problem SM-P in Theorem 7.2.

Proof. comes from the proof of Theorem 6.4. □

The formulation stated by Theorem 7.4 is the main subject of this work. Namely, we search for the solution $\{\boldsymbol{w}_n\}, \{\boldsymbol{w}_n^*\}$ in a similar manner as shown before by giving the stochastic version of radial return mapping algorithms.

7.3.1 Stochastic closest point projection

Following Sections 2.2.3, 2.3.2 and 5.2 we may formulate the closest point projection in the probabilistic setting.

Given an increment in displacement $\Delta \boldsymbol{u}_n \in \mathcal{U}$ compute $\Delta \boldsymbol{\varepsilon}_n = \boldsymbol{\nabla}_S \otimes \boldsymbol{I} \Delta \boldsymbol{u}_n$ and set the total strain $\Delta \boldsymbol{E} = (\Delta \boldsymbol{\varepsilon}_n, \boldsymbol{0})$. Further, taking the plastic flow rule:

$$\langle\langle \triangle \boldsymbol{E}_{pn}, \boldsymbol{T} - \boldsymbol{\Sigma}_n \rangle\rangle \leq 0, \quad \forall \boldsymbol{T} \in \mathscr{K} \tag{152}$$

one computes the increment of plastic strain as $\Delta \boldsymbol{E}_{p,n} = \boldsymbol{E}_{p,n} - \boldsymbol{E}_{p,n-1}$ i.e. $\Delta \boldsymbol{E}_{p,n} = \Delta \boldsymbol{E}_n + \boldsymbol{E}_{e,n-1} - \boldsymbol{E}_{e,n} = \boldsymbol{A}^{-1} : (\boldsymbol{\Sigma}^{trial} - \boldsymbol{\Sigma}_n)$ which gives the trial stress $\boldsymbol{\Sigma}^{trial} = \boldsymbol{A} : (\Delta \boldsymbol{E}_n + \boldsymbol{E}_{e,n-1})$ and hence the variational inequality:

$$\langle\langle \boldsymbol{\Sigma}_n, \mathbf{A}^{-1}(\boldsymbol{T} - \boldsymbol{\Sigma}_n) \rangle\rangle \geq \langle\langle \mathbf{A}^{-1} \boldsymbol{\Sigma}^{trial}, \boldsymbol{T} - \boldsymbol{\Sigma}_n \rangle\rangle, \tag{153}$$

to which corresponds the minimisation problem:

$$\boldsymbol{\Sigma}_n = \arg \min_{\boldsymbol{\Sigma} \in \mathscr{K}} \{ \frac{1}{2} \langle\langle \mathbf{A}^{-1}(\boldsymbol{\Sigma}^{trial} - \boldsymbol{\Sigma}), \boldsymbol{\Sigma}^{trial} - \boldsymbol{\Sigma} \rangle\rangle \}. \tag{154}$$

In other words, one obtains a stochastic minimisation problem to be solved for $\boldsymbol{\Sigma}_n$. This shows that one may use the closest-point-return algorithm, for which $\boldsymbol{\Sigma}_n$ is the projection of $\boldsymbol{\Sigma}^{trial}$ onto closed convex set $\mathscr{K}$ in the metric given by $\boldsymbol{A}^{-1}$, i.e. the norm $\langle\langle \boldsymbol{\Sigma}, \boldsymbol{A}^{-1} \boldsymbol{\Sigma} \rangle\rangle^{-1/2}$. The trial stress describes purely elastic behaviour. Thus, we employ the same operator split as in the deterministic counterpart. These operators are further called non-dissipative (purely elastic step) and dissipative (projection).

8 Conclusion

In this paper we have proposed an extension of the classical deterministic approach to the resolution of inelastic problems described by uncertain parameters or uncertain right-hand side. With the help of convex analysis and the theory of variational inequalities we have shown the mathematical similarity between the deterministic abstract variational formulation of the elastoplastic problem and its stochastic counterpart. Particularly, going from the description of the deterministic problem in one material point and definition of radial return map algorithm we have extended it to the abstract mixed variational formulation widely used in practice. Assuming the presence of uncertainty the abstract formulation is easily reformulated to the description of the stochastic generalised standard media with the help of some additional assumptions assuring the well-posedness of the problem. Further, this is specilisied to the quasi-static von Mises elastoplastic rate-independent evolution problem with linear isotropic hardening with the emphasis on the presence of uncertainty in the description of material parameters. Within one time-step of backward Euler discretization, we have shown that the problem may be reformulated as a minimisation for smooth convex functions on discrete tensor product subspaces, whose unique minimiser is obtained via the well-posed closest point projection method. To this end, we used a description in the language of non-dissipative and dissipative operators in order to formulate the stochastic closest point projection method.

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Notation

$\mathcal{G}$	computational domain
$\partial\mathcal{G}$	boundary
Γ_D	part of boundary with Dirichlet boundary condition
Γ_N	part of boundary with Neumann boundary condition
$\mathcal{T} = [0, T]$	time interval
$\boldsymbol{\sigma}$	stress tensor
$\boldsymbol{f}$	right hand side
$\boldsymbol{n}$	vector of normal
$\boldsymbol{u}$	displacement
$\boldsymbol{\varepsilon}$	strain tensor
∇_S	linear mapping between $\boldsymbol{\varepsilon}$ and $\boldsymbol{u}$
$(\cdot)_x$	variable in material point x
$\psi(\boldsymbol{\varepsilon})$	Helmholz free energy
$\boldsymbol{A}$	Constitutive 4th order tensor
$\boldsymbol{\varepsilon}_e$	elastic part of strain tensor $\boldsymbol{\varepsilon}$
$\boldsymbol{\varepsilon}_p$	plastic part of strain tensor $\boldsymbol{\varepsilon}$
$\mathcal{K}$	the elastic domain, a closed convex set containing the origin
$\dot{\boldsymbol{\varepsilon}}_p$	the plastic strain rate
$\mathcal{E}$	the space of strain rates $\dot{\boldsymbol{\varepsilon}}_p$
$\mathcal{R}$	the space of stresses $\boldsymbol{\sigma}$
$N_{\mathcal{K}}(\boldsymbol{\sigma})$	the normal cone at $\boldsymbol{\sigma} \in \mathcal{K}$
$\Psi_{\mathcal{K}}(\boldsymbol{\sigma})$	the indicator function
dom φ	the effective domain of function φ
$\partial\varphi(\boldsymbol{\sigma})$	subdifferential of a function φ
φ^*	dual function of a function φ
$\Psi^*_{\mathcal{K}}$	the support function of $\mathcal{K}$
$j(\dot{\boldsymbol{\varepsilon}}_p)$	the dissipation function
$\mathcal{K}^\infty$	the barrier cone of $\mathcal{K}$
$g_{\mathcal{K}}(\boldsymbol{\sigma})$	gauge functional of $\mathcal{K}$
$(\Psi^*_{\mathcal{K}})^o(\boldsymbol{\sigma})$	polar function of $\Psi^*_{\mathcal{K}}$
$\boldsymbol{\sigma}^{trial}$	trial stress
$(\cdot)_n$	variable in time step n of Euler backward method
$\mathcal{I}$	minimization functional
$\boldsymbol{E}_p$	generalized internal variables
$\boldsymbol{\Sigma}$	generalized stress (thermodynamic force)
ψ_{irr}	irreversible part of free energy
$\boldsymbol{H}$	tensor describing hardening of material
$\mathbf{A} = \mathrm{diag}(\boldsymbol{A}, \boldsymbol{H})$	general constitutive tensor

$\boldsymbol{\Sigma}^{trial}$	generalized trial stress
H_{iso}	isotropic hardening
H_{kin}	kinematic hardening
$\varsigma = -H_{kin}\boldsymbol{\varepsilon}_p$	backstress
$\zeta = -H_{iso}\nu$	conjugate force
$a(z,z)$	bilinear form
$(\cdot)^*$	the dual variable
L_p	Bochner-Lebesgue p-integrable functions
H^1	Sobolev space
a^p	bilinear form for perfect plasticity problem
a^g	bilinear form for linear hardening problem
a^d	bilinear form for linear hardening problem - dual form
$\boldsymbol{w}$	primal variable
$\boldsymbol{w}^*$	dual variable
$\mathcal{U} := H_0^1(\mathcal{G})$	the space of displacement
$\mathcal{F} := \mathcal{U}^*$	the dual space of displacement
$\mathcal{Q}$	the space of internal variables
$\mathcal{C}$	the space of conjugate forces
$\mathcal{Y}$	the space of generalized stress
$\mathcal{P}$	the space of generalized plastic strain
$\varphi(x,\omega)$	random field φ
$(S) = L_2(\Omega)$	the space of random variables of finite variance
$\mathscr{U} = \mathcal{U} \otimes (S)$	the space of stochastic solution
a^s	the bilinear form

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